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Professor A. FOWLER, F.R.S., PRESIDENT, in the Chair.

Ignatius Nicolas Dracopoli, F.R.G.S., Oak Hall, Bishop's Stortford, Herts,

was balloted for and duly elected a Fellow of the Society.

Dr. Vesto Melvin Slipher, Director of the Lowell Observatory, Flagstaff, Arizona, U.S.A.,

was balloted for and duly elected an Associate of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

L. J. Comrie, M.A., Auckland, New Zealand; temporary address, 32 The Avenue, Bruce Grove, London, N. 17 (proposed by T. H. Hudson); and

Professor J. S. Plaskett, Director of the Dominion Astronomical Observatory, Victoria, British Columbia (proposed by Sir Frank Dyson).

Sixty presents were announced as having been received since the last meeting, including, amongst others:

Stereograms of the Moon and Jupiter and of the Hydrogen and Nebulium images of the Orion Nebula, presented by Mr. J. H. Reynolds.

Results obtained from Seven Years' Observations made with the Cookson Floating Zenith-Telescope at the Royal Observatory, Greenwich. By H. Spencer Jones, M.A., B.Sc. (Plate 15.)

(*Communicated by the Astronomer Royal.*)

The Cookson floating zenith-telescope was loaned to the Royal Observatory in 1911 for a period of seven years by the Syndicate of the Cambridge University Observatory. This period has now expired, and it is the purpose of the present paper to present the principal results which have been obtained with the telescope during that period. The full details of the observations and reductions will appear as a Greenwich publication.

For particulars of the programme of observation, reference may be made to Professor Eddington's paper in *M.N.*, *R.A.S.*, 73, 605, 1913. It is sufficient to mention here that observations are made by the Talcott method, the stars being observed in pairs which are respectively north and south of the zenith and at nearly equal distances from it. The programme consists of 16 groups, each group containing 4 or 5 pairs which are photographed on one plate, and extending over about one hour of R.A. The groups are observed cyclically, so as to enable a satisfactory connection between the star-places to be obtained. The programme is arranged in such a way as to give aberration factors as large as possible.

The programme adopted has the following advantages:—

1. The star-trails whose distance apart is to be measured are close together, the mean zenith-distances with but few exceptions being less than 15′. They are therefore photographed in the centre of the field of the object-glass, so that no account need be taken of distortion, curvature of field, etc. Moreover, the use of a réseau is not necessary in measuring.

2. The trails are photographed on the meridian and are, therefore, parallel to one another. Errors of all adjustments are at a minimum; curvature corrections and other reductions are very simple to apply, and are small in amount; measurement is simplified.

3. The mean (algebraic) Z.D. of each group is very small, generally less than one minute of arc, so that corrections due to variation of scale are small in amount.

The Determination of the Scale and of its Temperature Coefficient.—These are determined by the observation of three groups of stars, containing in all 23 pairs of stars whose declinations differ by from $1\frac{1}{2}^\circ$ to 3°. These stars were all selected from Boss's *P.G.C.*, so that their places and proper motions are well determined. In observing these pairs the telescope is not reversed between the two stars of the pair, as in the case of the Talcott pairs. The distance between the two trails of one pair is, there-

fore, a measure of the declination difference. The scale plates were measured with the same micrometer as the latitude plates, and reduced with an assumed and approximate scale value of 5 mm. = 623″·05, which was known to be too small. The observed minus tabular value was obtained, and the difference for a declination difference of ten minutes was found. The mean difference for each plate was plotted against the temperature. The extreme values of the temperature were 26°·9 and 70°·8 F. The plotted points were practically on a straight line, indicating a linear dependence upon the temperature. There was some slight indication of the possible existence of a term depending upon the square of the temperature, although on theoretical grounds a term in dT/dt would be more likely to occur. Any such term, however, can safely be neglected in the reductions.

The deduced scale value is, therefore,

$$5 \text{ mm.} = 623''{\cdot}26 - 0''{\cdot}0041\ (T - 43^{\circ}{\cdot}5).$$

The latitude plates were reduced with a scale value 5 mm. = 623″·26, and the temperature corrections were applied to the mean of each plate.

The value of the temperature coefficient is in almost exact agreement with the value obtained by Cookson (*Mem. R.A.S.*, **60**, pt. ii., 101).

Measurement of Plates.—The plates were measured against a glass scale in a special micrometer. General division errors were found by the usual method of subdivision to avoid cumulative error, and were used in the determination of scale. In the latitude plates a small portion only of the scale was used. The division errors of this portion of the scale have been determined repeatedly and with special care.

Since the observations are differential, no correction for curvature is necessary, provided that the same length of each trail is measured. When a trail is defective and a portion only is measured, a suitable correction is applied to reduce it to the place it would have had if measured in the normal way. The sign of this correction depends upon the direction in which the plate is placed in the micrometer.

The micrometer measures the tangent of the angular distance from the plate-centre. When this is large it differs from the angle by a term depending upon the cube of the distance as in astrographic reductions. This correction has been applied to the scale plates but not to the latitude plates, in general, because for them it is small and also nearly constant for the same star. In a few instances in which the star-trails were displaced on the plate owing to an error in setting the telescope, the tangent correction was applied.

Each plate was measured direct and reverse. Decided systematic differences between the two measures were found for certain trails. The differences are not instrumental, because no difference was found when a silver réseau was measured direct and

reverse. Moreover, when the two measurers measured in directions opposite to those in which they normally measured, they obtained each other's results. The differences appear to be due to a difference of magnitude of the two star-trails measured, so that both observers set too far to the left (or right) on a faint trail compared with a bright. As the mean of the two measures is used, no errors are introduced by these systematic differences.

Instrumental Corrections, etc.—No corrections have been applied for collimation, level, and azimuth errors. These have been checked from time to time by visual observation and kept within narrow limits. There is no likelihood of any serious change in them remaining long undetected, as the trails would then look unsymmetrical.

The correction due to these errors is independent of the length of the trail. If the deviation due to the combined errors is c, the error of Z.D. is $\frac{1}{2}c^2 \tan \delta$, and if $c = 1'$, $\delta = 72°$, the correction is only $''{\cdot}026$. Moreover, high north stars are rare, so that no systematic effect on the aberration would be produced.

The errors have been reduced whenever they were found to amount to as much as $20''$. They remain very steady, and adjustments are rarely necessary and are probably necessitated by slight accidental disturbances of the instrument during cleaning, etc., which mainly affect the azimuth. The weakest feature in the design of the instrument is the method of supporting the annular trough: a more solid support would have been desirable.

Refraction.—It is not necessary to apply any correction for refraction. The refractions of the two stars north and south of the zenith which form one pair nearly balance; a mean refraction affects both the scale plates and the latitude plates alike: it is therefore included in the scale value and is automatically eliminated. Changes in the refraction constant would barely amount in an extreme case to $''{\cdot}01$ correction for an individual star-pair, and would be practically eliminated in the mean of a group.

Proper Motions.—Where possible, proper motions in declination were assumed from the catalogues of Boss or Groombridge. In the remaining stars proper motions were derived from the comparison of the A.G.C. position with the recent Greenwich observations, supplemented in a few instances by comparisons of two Greenwich astrographic plates with a difference in epoch of about twenty years.

Method of Reduction.—Preliminary values of the mean zenith distance of each pair of stars were assumed for the early observations. The mean precessions and proper motions for each pair were applied, and the assumed positions were tabulated for the beginning of each year from 1911·0 to 1919·0. Star corrections were computed, including the small nutation terms, and applied to the measures (proper motions not being included) to reduce them to the beginning of the year. The residuals obtained by subtracting the assumed values from the observed positions were tabulated, and the mean residual for each plate obtained.

To these mean residuals the following corrections were applied:—

(1) The fraction of the mean annual proper motions for the group, to reduce to the beginning of the year. If the group was incomplete, the mean proper motion for such pairs only as were measured was used.

(2) The temperature correction was applied to the mean Z.D. of the group, to reduce to the average temperature of 43°·5 F.

(3) In adopting assumed star-places, no endeavour was made to ensure that each star-pair in any given group would give—apart from accidental errors—the same value of the latitude variation. Provided that complete groups only had to be dealt with, this would have been immaterial, and the mean error would have merely entered into the group correction. A somewhat large proportion, however, of the groups have one or two pairs missing, the fainter stars not giving measurable trails when the definition is poor or through trails not being recorded owing to passing clouds. It therefore became necessary to apply a correction to reduce each pair of a group to the mean of that group. These corrections were determined from the mean discordances between the residuals given by the several pairs of a group observed on the same night. A correction for each pair was thus obtained, and the mean correction for the pairs measured was applied to each plate.

The residuals thus corrected were used in the subsequent discussion. As explained in the paper in *M.N.*, *R.A.S.*, **75**, 542, 1915, the observing programme contains 16 groups, observed in a cyclical manner, the year being divided into ten periods. The plates of a given group (the pth), which are observed during the qth period, give an equation of condition of the form,

$$z_p + t_q + \alpha_{pq}\mathrm{K} = \text{mean residual}$$

where z_p is the mean group correction to the assumed star-places for the pth group, t_q is the latitude variation during the qth period, α_{pq} is the mean aberration factor for this group and period, and 20″·47K is the correction to the assumed value, 20″·47, of the aberration constant. The weight of the equation is the weighted number of plates upon which it is based, plates with more than two trails having weight 1 and those with only two trails a weight $\frac{1}{2}$.

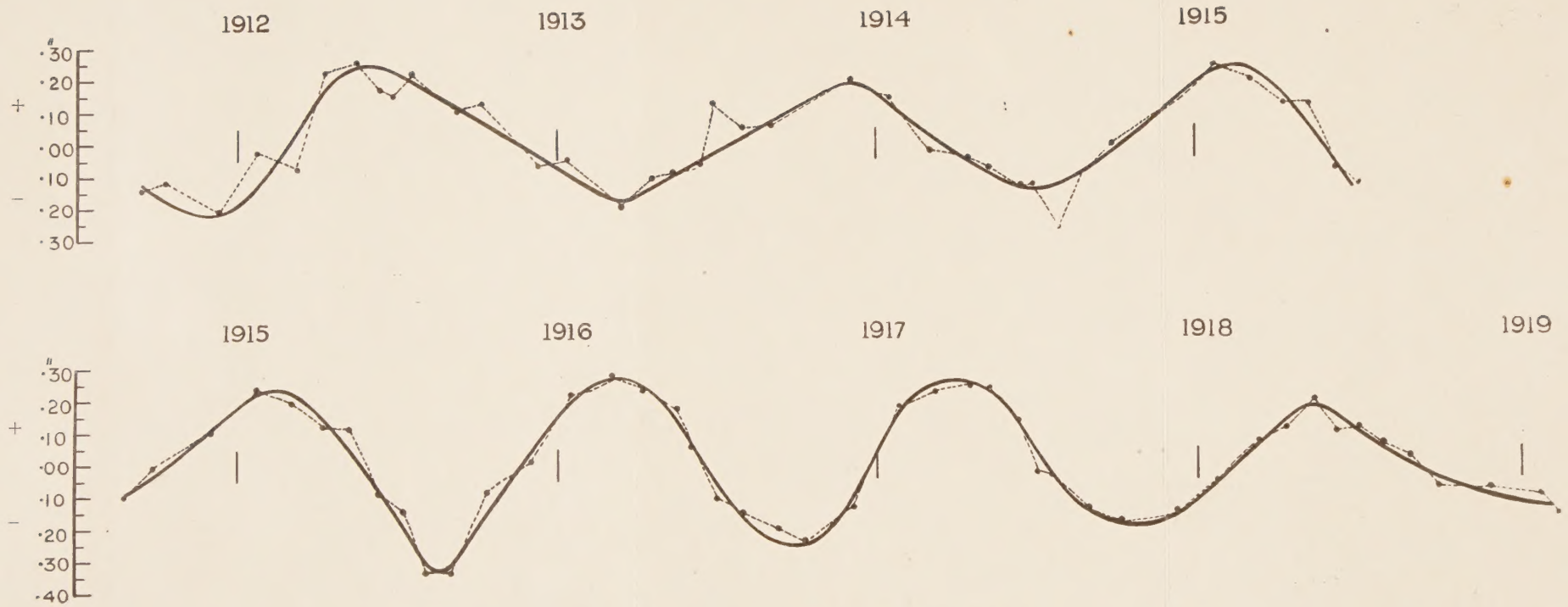
In the seven years there are 70 values of q and there are 16 p's. Normal equations for the 16 z's, 70 t's, and for K were constructed. The normal equations for the t's are simple, not involving more than 4 z's. Those for the z's may contain as many as 21 of the t's. The t's were therefore accurately substituted in the z normal equations in terms of the z's and K. 16 equations were thus obtained in terms of the 16 z's possessing the characteristics of a normal set of equations. These equations, however, are not independent, as only the differences of the z's can be determined. This is evident, because in the above equations

of condition all the z's can be increased by any arbitrary amount provided that the t's are decreased by the same amount, without the equation being altered. Writing the equations in determinantal form with a coefficient zero for any z which is absent, the sum of the coefficients in any horizontal or vertical line is zero. Similarly the sum of the constant and K terms is zero. These properties provided a useful check on the arithmetical accuracy. Moreover, they can be easily arranged to persist throughout the subsequent successive elimination of the z's by suitable arrangement of the method of elimination. Finally, the last z was put equal to zero, and the others determined. If the resulting latitude variations had been found to be unsymmetrical with regard to the zero-line, a correction would have been applied to all the z's of suitable amount. This was found not to be necessary. It may be remarked, however, that the application of any such correction would not have affected the finally deduced value of K, although it would have affected the deduced co-latitude of the observing hut.

The values of the t's were then obtained in terms of K and substituted, with the z values, in the normal equation for K, which contained the whole of the 86 other unknowns. In addition, and in order to provide some information as to the reliability of the deduced aberration correction, the group corrections provided by the whole series of observations were assumed to be accurate, and the normal equations for K provided by each separate year's observations were obtained and the resulting values of K calculated.

Results.

1. *Latitude Variation.*—In the following table are given the actual values of the latitude variation (in the sense mean – observed latitude) as deduced from the normal equations, together with the mean epoch of the period to which they refer. The term in K is not included in this table, for reasons referred to below. Values of the latitude variation for five periods subsequent to the completion of the exact period of seven years' observations have been included. These values, except where bracketed, are definitive, and replace the preliminary values which have from time to time been previously published in the *Monthly Notices*.



Latitude Variation at the Royal Observatory, Greenwich, 1911-1919.

TABLE I.

Latitude Variation at Greenwich, 1911·7–1919·1.

Date.	Lat. Variation.	Date.	Lat. Variation.
	″		″
1911·70	−0·140	1915·51	−0·142
·77	− ·116	·58	− ·336
·94	− ·207	·66	− ·337
1912·06	− ·023	·77	− ·089
·18	− ·070	·91	+ ·012
·27	+ ·227	1916·04	+ ·211
·37	+ ·257	·17	+ ·270
·44	+ ·175	·27	+ ·237
·48	+ ·154	·37	+ ·170
·54	+ ·220	·42	+ ·059
·68	+ ·104	·50	− ·106
·76	+ ·126	·58	− ·157
·94	− ·062	·69	− ·201
1913·03	− ·054	·77	− ·247
·20	− ·191	·92	− ·130
·30	− ·109	1917·07	+ ·178
·36	− ·094	·18	+ ·222
·45	− ·063	·29	+ ·244
·49	+ ·122	·35	+ ·233
·58	+ ·044	·44	+ ·130
·67	+ ·058	·50	− ·023
·76	+ ·116	·54	− ·035
·92	+ ·194	·66	− ·149
1914·04	+ ·142	·76	− ·184
·17	− ·024	·94	− ·155
·29	− ·049	1918·06	− ·054
·35	− ·077	·19	+ ·068
·45	− ·138	·27	+ ·102
·49	− ·132	·36	+ ·191
·57	− ·264	·43	+ ·099
·65	− ·091	·50	+ ·105
·74	− ·008	·57	+ ·058
·91	+ ·109	·66	+ ·012
1915·06	+ ·234	·74	− ·071
·17	+ ·191	·90	− ·084
·27	+ ·119	1919·06	(− ·106)
·35	+ ·114	·11	(− ·164)
·44	− ·083		

These results were plotted and a smooth curve drawn through them to represent them as closely as possible, consideration being paid to the relative weight of each value in the above table. From this curve the values given in Table II. were read off. The values in this table have the reverse sign to those given in Table I., being corrections applicable to observed declinations. They are, therefore, taken in the same sense as, and are directly comparable with, the x values of the International Latitude Service.

TABLE II.

Smoothed Values of Latitude Variation applicable to Observed Declinations.

Year. / Fraction of Year.	1911.	1912.	1913.	1914.	1915.	1916.	1917.	1918.
	″	″	″	″	″	″	″	″
·00	...	+·180	+·070	−·160	−·184	−·160	−·033	+·107
·05	...	+·132	+·102	−·124	−·216	−·208	−·122	+·065
·10	...	+·070	+·132	−·080	−·235	−·244	−·191	+·018
·15	...	+·004	+·160	−·042	−·233	−·266	−·228	−·027
·20	...	−·076	+·178	−·004	−·210	−·269	−·249	−·071
·25	...	−·156	+·162	+·028	−·170	−·255	−·254	−·116
·30	...	−·210	+·138	+·060	−·115	−·223	−·248	−·156
·35	...	−·242	+·110	+·090	−·055	−·172	−·221	−·180
·40	...	−·252	+·085	+·118	+·014	−·100	−·178	−·169
·45	...	−·246	+·060	+·136	+·087	−·020	−·108	−·136
·50	...	−·226	+·032	+·140	+·168	+·060	−·030	−·099
·55	...	−·200	+·005	+·134	+·250	+·131	+·045	−·062
·60	...	−·170	−·022	+·113	+·322	+·185	+·100	−·032
·65	...	−·142	−·050	+·087	+·320	+·222	+·139	−·001
·70	+·120	−·110	−·080	+·053	+·253	+·245	+·167	+·024
·75	+·155	−·080	−·108	+·018	+·178	+·250	+·183	+·048
·80	+·186	−·052	−·138	−·022	+·105	+·237	+·188	+·068
·85	+·205	−·022	−·166	−·060	+·038	+·204	+·185	+·080
·90	+·217	+·010	−·185	−·101	−·034	+·148	+·170	+·097
·95	+·210	+·040	−·185	−·144	−·098	+·060	+·147	+·108
1·00	+·180	+·070	−·160	−·184	−·160	−·033	+·107	+·120

2. *The Aberration Constant.*—The normal equation for K, after substituting the values of the latitude variations and the group corrections previously determined, reduced to

$$55{,}225\mathrm{K} = -74{\cdot}894,$$

which gives for the correction (20″·47K) to the assumed value 20″·470 of the aberration constant the value −″·0278. The final value of the aberration constant resulting from the seven years,

observations is therefore 20″·442, corresponding to a solar parallax of 8″·815. The value obtained is therefore very close to the old value of Struve.

In order to obtain some information as to the weight to be attached to this value, without a detailed investigation of probable errors, the corrections resulting from each single year's observations were determined. The normal equations and aberration corrections are given in Table III.

TABLE III.

Year.	Normal Equation for K.	Correction to 20″·470.
1	5·493K = +12·424	+0·0463
2	4·567 −13·738	− ·0616
3	8·441 −10·270	− ·0249
4	9·615 −33·958	− ·0723
5	8·461 −32·408	− ·0784
6	8·859 +20·882	+ ·0483
7	9·789 17·826	− ·0373

These results show a much larger variation than had been anticipated *a priori*, and indicate the presence of large accidental errors. From the discordances between them, the probable error of the value obtained for the correction to the aberration constant from one year's observations is about ±″·035 or for the mean of seven years it is ±″·013. The corresponding probable error of the value 8″·815 of the solar parallax is ±″·006. It is of interest to note that the final value obtained by Hinks from observations of Eros was 8″·806, ±″·004, and the two values agree, therefore, within the limits allowable by the specified probable errors.

Whilst the observations would appear to indicate a slight negative correction, too much emphasis is not laid upon the actual value obtained, and accordingly, in adopting final values for the latitude variation, the term in K has been neglected. Had the value 20″·470 been adopted for the aberration and the telescope used merely for the determination of latitude variation, the values finally obtained for the latitudes would have been those which we have adopted.

In order to obtain some information as to the extent to which the deduced value of the aberration correction is dependent upon accidental errors, the computation was repeated, assuming the values obtained for the group corrections to be accurate, but excluding all plates whose residuals, after taking out latitude variation and group corrections, exceeded ±″·200. The values for the several years showed the same general features, the signs of the corrections remaining unaltered, but the magnitudes in all but one instance being reduced. The values were +″·034, −″·036, −″·030, −″·038, −″·035, +″·033, −″·026. These values give a correction −″·015, corresponding to an aberration constant of 20″·455 ± 0″·009, or a value of the solar parallax of 8″·810 ± 0″·004.

General Remarks on Results.—An examination of the plate accompanying this paper, which gives the latitude results actually obtained, together with their representation by a smooth curve, will show that only in a few exceptional cases does the observed variation deviate from the smooth curve by more than 0″·05. The smoothness of the actual results is such that the probable error of each latitude determination can fairly be stated as not exceeding ±0″·015. Thus, although the programme of observation was not designed primarily for the determination of latitude variation, it is evident that it is eminently suitable for this purpose. That it has enabled latitude variation at Greenwich to be obtained so accurately has proved of great value, as the results have been available for use a considerable time before the corresponding international results have been published. The values determined with the Cookson telescope have been used in the Greenwich meridian reductions.

The accuracy with which the latitude variation was being determined had led to the hope that the aberration constant would also be obtained with a small probable error. The results have once again proved that the determination of the aberration constant is a matter of far greater difficulty than the determination of latitude variation. The accuracy of the latter is dependent very largely upon the multiplication of plates, which enables satisfactory group corrections to be obtained and which reduces the mean accidental error. For latitude work, therefore, observations at any time during the night are of equal value. For aberration determination, however, observations should be made as nearly as possible at 6 o'clock in the evening and in the morning, in order to obtain as large a value as possible of the aberration factor. Each plate carries a weight proportional to its factor. The plates obtained during the winter have, therefore, a much greater effective weight than those obtained in the summer. In the middle of summer the plates must necessarily be obtained near midnight, and carry very little weight for the determination of aberration. Unfortunately, the plates obtained during the winter months are not only fewer but their quality is much poorer than those obtained during the summer months. On the average, the summer plates show trails of stars at least one-half magnitude fainter than the faintest stars which are recorded during the winter. A far greater proportion of the winter plates therefore have reduced weights through one or two pairs of trails of the fainter stars being too faint to be measurable.

The tendency on the part of the measurers has been to measure whatever trails have been visible. When a lengthy spell of cloudy weather has prevailed and only a few poor plates have been obtained during the interval, it has been necessary to include these plates in order to maintain adequate group corrections. Where plates have been numerous, a few bad plates have been rejected, but it might have been advisable to reject much more freely. It is difficult, however, to reject plates once measured without per-

sonal bias coming into play. In future, therefore, all plates not considered suitable for accurate measurement when viewed in the micrometer, will be forthwith rejected.

The greatest disadvantage under which the photographic method labours is that the range of magnitude available for use is very limited. Stars north of the zenith can be used which are of a fainter magnitude than stars south of the zenith. On the average, however, it may be stated that stars brighter than $5^{m}\cdot0$ (photographic) are too bright, and that stars fainter than $6^{m}\cdot4$ are too faint. A bright star gives a broad black trail upon which it is difficult to set the two micrometer wires accurately. Moreover, such stars give large personal differences in measuring direct and reverse, which would probably prove troublesome if several measurers were used, since their personalities might conceivably differ considerably. With faint stars, too large a proportion of trails are found to be fainter than is desirable for accurate measurement. The original catalogue also contained a few double stars, and it is found that these are very objectionable in the micrometer if their separation is less than about 15″.

In the light of present experience it seems that a somewhat larger aperture for the object glass is desirable in order that stars down to, say, $7^{m}\cdot0$ (photographic) can be obtained with facility. A much greater choice of stars would then be available. The present choice of stars is unduly restricted, in view of the number of conditions which are required to be satisfied. These are, that (i) the two stars of each pair must be at equal distances north and south of the zenith, within about 15′; (ii) their right ascensions must differ by not less than 3 minutes in order to allow the telescope to settle down after rotation; (iii) the right ascensions of the second star of one pair and the first of the succeeding must differ by not less than 4 minutes; (iv) one hour of R.A. must contain at least four star-pairs with sufficient time left for changing to the next plate; (v) the mean Z.D. of all pairs in one group must not exceed about 120″, and must be so adjusted that the effect of precession throughout the seven-year period will not remove it beyond this limit; (vi) the photographic magnitudes of each star should be within the limits $5^{m}\cdot0$ and $6^{m}\cdot4$.

With a larger aperture object-glass it might be advisable to use a suitably designed stop to reduce the aperture in the case of the brighter stars.

Alternatively, a lens of the present aperture would give stars of the required faintness if it were a doublet instead of a triplet. The triplet type of lens was decided upon by Cookson in order to obtain the flat field which was essential for his method of using the telescope. With the programme decided upon at Greenwich, the centre of the field only is used. A doublet lens would therefore suffice, and on account of the considerably diminished absorption in the glass would prove an advantage.

Accuracy of Measures, etc.—The following results give an

indication of the probable error of the results derivable from a single plate and of the way in which the error is made up :—

1. The mean discordance between two measures of one pair of star-trails, derived from an examination of all pairs which give no systematic differences when measured direct and reverse, is $\pm 0''\cdot 11$. From this it follows that the probable error of the mean residual from one plate, arising from errors of measurement only, is $\pm 0''\cdot 023$.

2. The mean discordance between two pairs of trails on the same plate, obtained from the comparison of all pairs which required the same correction to reduce them to the mean of the plate, is $\pm 0''\cdot 214$. This corresponds to a probable error for the mean residual of one plate of $\pm 0''\cdot 045$. This will consist partly of errors of measurement and partly of other errors of an accidental nature due to varying instrumental, atmospheric, or other causes ; such errors would result in two star pairs giving different latitudes even if accurately measured. Separating the errors of measurement, we have for the "error of the star-trails" $\pm 0''\cdot 039$.

3. Separating the latitude variation and group corrections from the plate residuals and taking every instance of two plates being observed consecutively, we find that the mean discordance between two *consecutive* plates is $\pm 0''\cdot 099$, which would give the probable error of a single plate as $\pm 0''\cdot 059$. We have, however, first found that the probable error of a single plate, in so far as *internal* errors are concerned, is $\pm 0''\cdot 045$. The excess must be due to the totality of errors which affect consecutive plates accidentally, and to errors in the assumed latitudes. These errors therefore amount to $\pm 0''\cdot 038$.

4. The arithmetic mean of all the mean residuals used in the differencing in the preceding section is $0''\cdot 091$. This is in agreement with the mean plate residuals for all the plates used, viz. $0''\cdot 092$, and corresponds to a total probable error for one plate of $\pm 0''\cdot 078$. The difference between this value and the value $\pm 0''\cdot 059$ previously found is due to causes which affect consecutive plates equally. Such causes were denoted by Professor Eddington by "error of the night," as he obtained some evidence that they persist through the night. This error is of amount $\pm 0''\cdot 051$.

We have then the following summary of the various errors :—

Error of measurement . . .	$\pm ''\cdot 023$
,, the star-images . . .	$\pm \cdot 039$
,, the plate	$\pm \cdot 038$
,, the night	$\pm \cdot 051$
Resultant	$\pm \cdot 078$

Investigation into the cause of the "error of the night" is being carried out, in order to ascertain whether its effects are accidental or systematic in nature.

Each value of the latitude variation depends on the average upon the mean of about 20 plates. It follows that the probable error of each value should be about $\pm 0''\cdot 017$, which is in good

agreement with the estimate made above from inspection of the plotted results.

Future Observations.—The results which have been obtained have provided a reliable determination of latitude variation and a satisfactory connection between the star-places. Several spells of bad weather during the winter months have been largely responsible for the somewhat large probable error of the correction to the aberration constant. The experience gained during the past seven years confirms earlier opinions as to the suitability of the Cookson telescope for this work; it is also probable that the arrangement of the observations could not be improved.

The Syndics of the Cambridge Observatory have kindly loaned the telescope to the Royal Observatory for a further period of seven years. The same programme of observations is being continued; a few necessary alterations in the star list have been made, the brightest and faintest of the stars having been replaced by stars of intermediate brightness and the mean Z.D.'s suitably adjusted.

The second period of seven years is therefore being commenced with the advantage of having a programme in which the large majority of the stars have well-determined places and proper motions. There is consequently not so much necessity to include scattered poor plates, which were previously incorporated so as to obtain adequate group connections. No trails not considered suitable for accurate measurement will be measured in the micrometer. In this way it is believed that the accidental errors will be considerably reduced, and it is hoped that the final aberration correction will have a probable error of sufficient smallness for considerable weight to be given to it in any discussion as to the most probable value of the aberration constant.

The Eruptive Prominence of 1919 *May* 29. By W. Moss. (Plate 16.)

(*Communicated by the Director, Solar Physics Observatory, Cambridge.*)

On May. 28 photographs were obtained on the Kensington spectroheliograph showing a large prominence in the S.E. quadrant extending from lat. 42° S. to lat. 20° S. and 2′·2 high. There was also a smaller prominence at lat. 7° to 17° N.

On the following morning the first photograph of the prominence showed such developments that, in view of the total solar eclipse on that day, a succession of plates were taken throughout the day. A collection of most of them is given in Plate 16.

A photograph taken at $29^{d}\ 19^{h}\ 16^{m}$ showed that the prominence was still in existence, though much reduced.

On the 28th, at $2^{h}\ 10^{m}$, the prominence suggests a long roll of smoke cloud resting on the limb, and fairly full of detail.

Next morning, at $20^{h}\ 35^{m}$, the prominence has risen above the

limb, and is best described as a strong upcast at lat. 40° S., with a cloud extending from the top towards the equator. The floor of the cloud is furthest from the limb at its northern tip at lat. 13° S., where it is 2″·5 above the limb. The maximum height of the roof is 3″·7 over lat. 25°. The floor is undulatory and brighter than the body. From brighter parts over lats. 22° and 27° wavy streamers connect it to the limb at lower latitudes. The upcast is about 5° wide at the base, and tapers upwards. The tilt of the sun's axis is only 1°, so that it does not introduce serious errors into the latitudes of the higher points.

By $21^h\ 20^m$ the floor has straightened somewhat, and the tip lengthened.

By $22^h\ 20^m$ the outline of the northern half of the cloud has hardened, notably on the under surface; the southern half is more diffuse. The streamers down from the floor are longer and, leaving the floor at lats. 27° and 22°, they reach the limb at lats. 23° and 16°. The upper part of the upcast is now somewhat stronger, and from the northern side filaments reach to the southern descending streamer.

By $23^h\ 21^m$ the tip of the cloud has reached a point over lat. 8°. The upcast is broader. The southern streamer has disappeared and the northern one become more diffuse and has moved a little northward. The definition on this negative is not good.

The next negative was taken at $29^d\ 1^h\ 12^m$, when the definition had improved. The northern tip of the cloud has split and the bottom part has extended forward and downward until it reaches the limb at the equator. The space under the cloud is filling with radiant material. The split at the base of the upcast, which can be recognised in earlier negatives, is now very marked. The floor of the cloud is now 2″·2 above the limb, and the maximum height of the roof is 4″·0 over lat. 25°.

By $1^h\ 51^m$ the split tip has widened. The details under the arch can now be recognised as intersecting arcs.

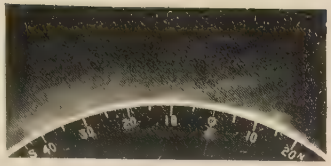
By $2^h\ 29^m$ the streamer from the split end of the cloud, which reaches the equator, has itself split just above the limb. The weaker part still makes contact at the equator, and a new and much stronger streamer runs down to lat. 4° N. The disruption of the upcast and its connection with the cloud is proceeding..

By $3^h\ 5^m$ the streamer reaching the equatorial region has weakened.

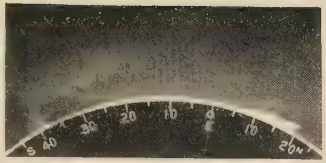
By $3^h\ 59^m$ the disconnection of the cloud from the upcast seems to be complete. The split in the northern tip is widening and the streamers weakening.

By $4^h\ 19^m$ the maximum height is 5″·3 above the limb, the height of the floor about 3″·0. The cloud is beginning to float loose at both ends, and on the equator there is a very intense patch 3° wide and 0″·8 high.

By $4^h\ 44^m$ the patch on the equator has grown much less. The separation of the cloud from the upcast and the down streamers is complete.



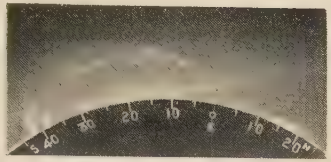
1. 1919 May 28d 2h 10m.



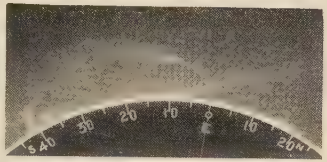
2. 1919 May 28d 20h 35m.

3. 1919 May 28d 21h 20m.

4. 1919 May 28d 22h 20m.

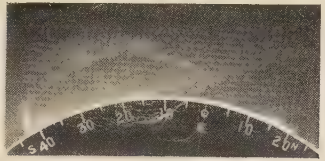


5. 1919 May 29d 1h 12m.

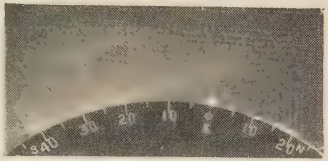


6. 1919 May 29d 1h 51m.

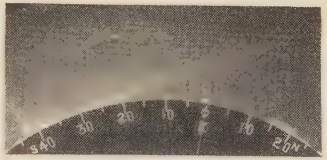
7. 1919 May 29d 2h 29m.



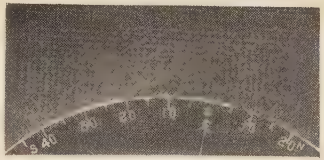
8. 1919 May 29d 3h 5m.



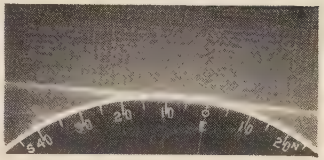
9. 1919 May 29d 4h 19m.



10. 1919 May 29d 4h 44m.



11. 1919 May 29d 5h 53m.



12. 1919 May 29d 19h 46m.

The last photograph before sunset was taken at $5^h\ 53^m$, when the sun was distinctly yellowish. The negative is decidedly weak. The upcast is easily visible, and it has tilted over towards the south. Of the cloud, the negative shows only a very faint patch over lat. 10° to 20° and at a height of 5′·3 and one or two other faint wisps. (These are not visible in the reproduction.)

On a negative obtained next morning at $19^h\ 46^m$ and under better atmospheric conditions, only the base of the upcast is visible.

The times given above are those when the primary slit of the instrument was tangential to the sun's E. limb. The tangent is marked by a streak of increased diffuse light on the background. The maximum time of traverse across the prominence was about two minutes.

The rate of extension towards the equator, parallel to the solar surface, was fairly regular. The extension between $28^d\ 20^h\ 35^m$ and $29^d\ 4^h\ 19^m$ amounts to 22°, or roughly 190,000 miles. Between $28^d\ 23^h\ 21^m$ and $29^d\ 1^h\ 12^m$ there is, however, the downward projection of the end of the streamer to the equator to be considered, and during these two hours the growth of the streamer amounted to about 90,000 miles.

The rate of rise of the roof of the cloud over lat. 20°, between $28^d\ 20^h\ 35^m$ and $29^d\ 3^h\ 59^m$, was small and moderately regular. At about $29^d\ 3^h\ 59^m$, however, the cloud is breaking loose at the northern end, and by $29^d\ 4^h\ 44^m$ (*i.e.* 45 minutes later) the roof over lat. 20° has lifted from 4′·0 high to 5′·5 high, giving a velocity upwards of approximately 55,000 miles per hour.

The southern end of the cloud broke loose earlier, and does not show such an abrupt change in the rate of rise.

The maximum length is about 48°, or 362,000 miles. The maximum height is 5·6 minutes, or 150,000 miles at $4^h\ 44^m$.

Several of the negatives were made by Mr. C. P. Butler, who is also responsible for the orientation of all the negatives.

1919 *June* 12.

The Spectrum of Nova Aquilæ, 1918 *August* 23 to *October* 23.

By the Rev. A. L. Cortie, S.J.

The present paper discusses photographic spectra of Nova Aquilæ obtained with the Thorp prismatic camera on the dates 1918 August 23, September 25, October 4, and October 23, and with the Whitelow prismatic camera on August 23 and October 23. The former instrument has an objective prism of 3 inches aperture and 22·5° angle, mounted over a 3-inch achromatic lens made by Cooke of York. The focal length is 50 inches. The latter instrument consists of a prism of 19° angle and 6 inches edge in combination with a Dallmeyer portrait lens of 6 inches aperture and 30 inches focal length.

The last paper of this series dealt with the spectrum of the nova on 1918 July 29, when the spectrum was mainly of the

γ Orionis type, with, however, a well-marked accompanying spectrum of bright lines of the α Cygni type. The magnitude of the star on that date was approximately 4·0. In the interval between July 29 and October 23 the star steadily declined in magnitude. According to the photographic measurements of Mr. Hollis * at Greenwich, the decline was from 4·02 to 5·55 magnitude, measured on October 22, a range of 1·53 magnitudes. The visual determinations of Dr. Rambaut † in the same interval make the decline in magnitude 1·08, from 3·89 magnitude on July 29 to 4·97 on October 22. Father Hagen's ‡ values for July 28 and October 22 are 3·8 and 5·3 magnitudes, the decline of 1·5 agreeing closely with the value deduced from the observations of Mr. Hollis. In the interval August 23 to October 23, covered by the spectra discussed in this paper, the nova fell 0·98 magnitude according to Mr. Hollis, 0·15 according to Dr. Rambaut, and 0·5 according to Father Hagen.

During this interval the light of the star in the photographed spectra was mainly concentrated in the radiations at or about λ 5007, H_β, λ 4640, H_γ, and H_δ, but the distribution of light in these radiations varied considerably. All the spectra were taken on Ilford Monarch plates, and the same developer, rytol, was used throughout. Although they all show the green radiation λ 5007, it is evident that the order of intensity of the several radiations would have differed had isochromatic plates been employed. The several radiations can be placed in the following approximate order on the successive dates:—

Thorp Camera.				Whitelow Camera.	
Aug. 23.	Sept. 25.	Oct. 4.	Oct. 23.	Aug. 23.	Oct. 23.
H_γ.	H_γ.	H_γ.	H_γ.	H_γ.	H_γ.
4640	5007	4640	...	4640	H_δ
H_β)	4640	5007	...	H_δ	4640
H_δ)	H_β	H_β	...	H_β	5007
5007	H_δ	H_δ	...	5007	H_β

On October 23 the trail with the Thorp camera was too fast, and the plate shows only one radiation very distinctly, namely, that about H_γ.

In setting the spectrum for the trail on September 25, it was noticed that by far the greater part of the light was concentrated in one intense radiation about H_γ. On October 4, although this radiation was the most intense, the other radiations were not relatively so weak as on September 25. On October 4 the spectrum was also observed visually with a McClean star spectroscope used as an eye-piece with the 15-inch equatorial. The red radiation H_α was visible, but very greatly reduced in brilliancy from its appearance in June and July. Also a bright yellow radiation, probably D_3, was seen. But by far the strongest radiation visually was

* *Monthly Notices, R.A.S.*, 79, 36. † *Ibid.*, 79, 108.
‡ *Ibid.*, 79, 168.

λ 5007, which was very bright. The same ray was also very prominent visually on October 23. According to the observations of Professor Barnard,* since 1918 October 5 the star has appeared as a planetary disc which was apparently growing. These visual observations would seem to indicate that the star had reached the stage of a planetary nebula, a conclusion which is borne out by the spectroscopic observations.

The Radiations about H_γ.—What is here termed the H_γ radiation includes not only the three components of the H_γ hydrogen radiation, but the nebula line λ 4363, which is coincident with the less refrangible component of H_γ. Also it includes a bright band still less refrangible, and having its edge about λ 4380, which may be due to the helium line λ 4388. It is this congeries of radiations which constituted the great brightness in the spectrum about the position of H_γ. The more refrangible component of the triple H_γ radiation has a mean wave-length λ 4322. This latter component gradually waned in intensity, and the maximum of brightness shifted towards the less refrangible side of the total radiation. This shows that the nebula line λ 4363 was becoming more prominent.

On August 23 there was much structure in this broad H_γ band, several sharp bright lines being observed in the three component bands. There was also a clear separation between the band λ 4380 and the edge λ 4363. But on September 25 the bright radiation was uninterrupted by any dark spaces from λ 4380 to λ 4334. There was at this position a marked dark space or break in the radiation, and the more refrangible component, about λ 4320, was much reduced in intensity. On August 23 in the Whitelow camera the nebula line λ 3869, on the more refrangible component of H_ζ, was observed to be strengthened in intensity. It was brighter again on October 23 in the same instrument. These indications of the great relative strength in the three radiations λ 5007, λ 4363, and λ 3869 show that by October 23 the spectrum was mainly of the type of that of a planetary nebula.

The following table gives the wave-lengths of the bright edges and bright maxima in the component bands of H_γ, on the several dates, compared with those for July 29 :—

H_γ *band.*

July 29.	I.	Aug. 23.	I.	Sept. 25.	I.	Oct. 4.	I.	Oct. 23.	I.
λ 4317		λ 4319		λ 4316		λ 4316		λ 4318	
24 21	9	23 24	7	26 21	4	27 22	2	27 23	1
28		28							
38		36		34	8	34	9	35	
44 41	10	40 40	9	41 39	9	41 40	9	41 40	9
		44		44	9	46	9	45	
56		55	9	50	8	49		52	
62 59	10	57 61	10	54 61	10	64 57	10	68 60	10
		66	9	62	9				

* *Astrophysical Journal*, **49**, 149.

These separate measures, the means of at least three settings for each, and made independently, are very consistent. They show the maintenance of the triple structure of the H_γ band, and the gradual falling in intensity in the more refrangible component.

The Radiation λ 4380.—On June 15 a bright band was measured with edges at λ 4378 and λ 4388, mean wave-length λ 4383, and intensity 2. This band is a companion to, and is less refrangible than, the λ 4363 component of H_γ. It was attributed to helium λ 4388, as one edge coincided with that wave-length. In Nova Persei and in Nova Aurigæ, as measured by Father Sidgreaves, the mean wave-lengths of the band were at λ 4380 and λ 4382. In Nova Geminorum II. my own measures gave λ 4373 and λ 4380 for the edges of the band, and mean wave-lengths λ 4372 in 1912, and λ 4375 in 1913. This last position is practically coincident with λ 4376, the value derived from the Madrid photographs of the same nova.

The following are the measures since and including July 29:—

Band about λ 4380.

	Thorp Camera.												Whitelow Camera.			
	July 29.		Aug. 23.			Sept. 25.			Oct. 4.			Oct. 23.			Aug. 23.		Oct. 23.	
	I.		I.			I.			I.			I.			I.		I.	
λ 4373	79	9	4375	80	8	4367	74	8	4370	4381	6	4373	4382	8	5	...	...	
85			80			73										4385	7	4387 10
			85			79												

In the Whitelow camera with its small dispersion, a very small difference in the micrometer setting makes a marked difference in wave-length. The line λ 4369 is found in O-type stars. The helium line λ 4388 is probably represented by the less refrangible edge of the band. But the origin of the radiation about λ 4376 is at present obscure. The wave-lengths for September 25 are anomalous, as the composite radiation including this band extended continuously from λ 4367 to λ 4334. The readings at λ 4374 and λ 4379 are bright maxima in the band.

The Width of the Hydrogen Bands.—The following table shows the width of the hydrogen bands H_β, H_γ, H_δ, H_ϵ in Angstrom units, as measured independently on the several dates set down:—

Width of the Hydrogen Bands in Angstrom Units.

	H_β.	H_γ.	H_δ.	H_ϵ.
June 15	59	50	42	40
July 29	52	45	43	41
Aug. 23	48	47	44	43
Sept. 25	55	46	40	
Oct. 4	50	48	48	
Oct. 23	59	50	42	40

It is evident, therefore, that the width of the hydrogen bands remained practically constant. Such relatively small differences in width as are shown on the plates are most probably due to different rates of trail and consequent differences in exposure.

The H_β and H_δ bands were more uniform than H_γ, and showed the same structure in the period August 23 to October 23. The radiations H_ϵ and H_ζ were faint on the plates obtained with the larger dispersion of the Thorp camera after August 23, but were quite evident on the smaller dispersion of the Whitelow camera. The conclusion to be deduced is that the width of the hydrogen radiations remained constant during more than five months, covered by the observations June 15 to October 23. If, as there is reason to think, the outer components of these radiations represent line-of-sight velocities of recession and approach, then we must also conclude that the velocities have remained constant. But that velocities of the order of 1300 and 1100 km. per second should have been maintained for five months seems to be most unlikely.

The Band λ 4690.—Among the most prominent lines in O-type or Wolf-Rayet stars is a line at λ 4688. It does not appear in my measures of the bright-band spectrum on June 15, but a band corresponding to this wave-length, within the limits of error, was prominent in the spectrum of the nova on July 29 and following dates.

The measures give mean wave-lengths July 29, λ 4687; August 23, λ 4692; September 25, λ 4692; October 23, λ 4692. The appearance of this radiation was another sign that the spectrum of the nova had changed into that corresponding to a planetary nebula.

The α Cygni Type of Spectrum.—The photographs of the spectrum, taken with the Thorp camera on August 23, September 25, and October 4, were very carefully compared with that taken on July 29. By setting the micrometer thread at the readings of the several bands and lines on July 29, and in many cases by making independent measures, it was found that the α Cygni type of spectrum of bright bands was still present. It was very much reduced in intensity, some of the lines being barely visible.

In the spectrum of August 23 and October 23, taken with the Whitelow camera, there is a well-marked line at λ 4296. This is most probably the enhanced titanium line λ 4294, which is strong in α Cygni but is not found in γ Orionis.

The Principal Lines and Bands on August 23 and October 23.—The following table gives the measures obtained on the photographs taken on these dates with the Whitelow camera. The advantage of the small dispersion, only 8·60 mm., for the range of spectrum λ 3836 to λ 5367, is that it concentrates the light of the principal radiations and makes them more easily recognisable. The obvious disadvantage is that a small error in setting the micrometer wire gives an error of 2 or 3 Angstrom units for a difference of only ·01 mm.

August 23.		October 23.		
Wave-length.	Intensity.	Wave-length.	Intensity.	Origin.
3836	3	...	...	H_η
69	3	3869	4	Nebula line
89	3	89	3	H_ζ
3970	5	70	3	H_ϵ
...	...	4008	1	He λ 4009
...	...	25	1	He λ 4026
4102	8	4102	5	H_δ
...	...	41	1	He λ 4143
73	1	73	1	He λ 4170
95	1	95	1	O λ 4190 ?
...	...	4249	1	
4257	1	60	0	O λ 4255, Nebula λ 4265
...	...	86	0	Cr * λ 4285
96	1	96	1	Ti * λ 4294
...	...	4328	6	$H_{\gamma 2}$ component
4342	10	42	10	H_γ
65	10	65	10	$H_{\gamma 1}$ component, Nebula λ 4363
85	7	87	10	He λ 4388
4428	1	...	...	λ 4427, July 29
...	...	4434	0	He λ 4438
65	1	69	0	He λ 4470
...	...	4507	1	λ 4507, July 29
4512	1	...	...	λ 4511, July 29
34	1	28	1	N λ 4531
51	1	57	1	Si* λ 4553, γ Orionis
...	...	4609	1	
4649	8			
40	8	40	6	Nebula λ 4640
...	..	75	6	O λ 4676
...	...	4710	6	Nebula λ 4686, He λ 4713
4747	2	41	2	Nebula λ 4744
...	...	63	2	
83	1	87	2	O stars λ 4787
4862	4	4862	3	H_β
4959	2	4959	1	Nebula
5006	4	5007	4	Nebula
...	...	87	0	Dark line, Nova Aurigæ λ 5085
5100	1			
78	1	5172	1	O λ 5176, Mg. b. group
5286	1	5289	0	Dark line, Nova Aurigæ λ 5286
5367	1	5367	0	Bright line, Nova Aurigæ λ 5364

Of the 41 lines in this list, 6 are due to hydrogen, 7 to nebulium, 7 to helium, 3 to oxygen, and 1 each to nitrogen, chromium, titanium, and silicon. The spectrum on these dates was therefore still of the γ Orionis type, but predominantly of the type belonging to a planetary nebula.

Summary.

1. The great relative strength of the chief nebula lines, and the presence of lines of O-type stars, shows that after August 23 the predominant type of spectrum of the nova was that of a planetary nebula.

2. The number of lines of helium, and the presence of lines of oxygen and nitrogen, indicate that the γ Orionis type of spectrum was still present until October 23.

3. The α Cygni type of spectrum was faintly maintained until at least October 4, and one spark line of titanium peculiar to it is found on October 23.

4. The width of the hydrogen bands has remained practically constant for more than five months.

Stonyhurst College Observatory:
1919 *June* 7.

The Distribution of Hydrogen and Nebulium in the Orion Nebula.

By J. H. Reynolds.

As long ago as 1899 Keeler made an exposure with the Crossley reflector of the Lick Observatory on the Orion Nebula, using an isochromatic plate and a green screen, thus obtaining an image which showed approximately the distribution of nebulium in the nebula. He compared the result with unscreened images on ordinary fast plates, and was able to confirm the conclusion already drawn from spectroscopic observations that nebulium was concentrated in the Huyghenian region and its neighbourhood.* In 1905 Hartmann published a paper in the *Ap. J.* dealing principally with a monochromatic image of the nebula in the radiation at λ3727, which showed a large extension beyond the nebulium region, and he concluded that this radiation was responsible for the great photographic activity of the outer regions of the nebula, which Keeler had attributed to hydrogen.

He also photographed the nebula with a picric acid screen (transmitting $\lambda\lambda$5007, 4958, and H_β) and an isochromatic plate to get a comparative nebulium image, and in addition he attempted some photographs with a quinine cobalt screen transmitting the range from $\lambda\lambda$4740 to 3880, which includes the hydrogen series from H_γ.† His exposures with the latter were, however, too short to give sufficient extension and density to judge of the distribution of hydrogen in the nebula.

* *Ap. J.*, 9.

† This range is given in the *Ap. J.* as $\lambda\lambda$3740 to 3880—obviously a misprint.

Early in the present year I undertook a similar series of photographs with the 28-inch reflector in the nebulium and hydrogen radiations, but a different method was adopted for the hydrogen images. An æsculine screen was used with Imperial Flashlight plates cutting out the ultra violet beyond λ3850 altogether, and weakening considerably the range between λλ3850 and 4000. To ensure that the æsculine screen was capable of effecting this satisfactorily, the photographic spectrum of a similar screen placed in front of a strontium arc reaching to λ2500 on the unscreened plate was supplied to me by the Wratten Division of Kodak Ltd., which was conclusive as to its efficiency. This screen allowed the longer wave-lengths from λ4000 to pass with very little loss of intensity, and proved to be very suitable for the purpose. The effective photographic range included H_γ, H_δ, and H_ϵ, Nu 4363 and the faint helium radiations at λ4472 and 4390. Under the conditions, none of the radiations except the hydrogen need be taken into account, as they were too faint to affect the result substantially. The main point was to exclude the very active radiation at λ3727, and this has evidently been done. Two other methods of dealing with hydrogen radiations were tried: one of these was a specifically blue screen transmitting H_γ and H_δ, the other a yellow screen cutting out wave-lengths shorter than λ5100 in conjunction with a red sensitive plate.

On examining the nebula through a piece of orange glass a faint orange image of the Huyghenian region was seen which may safely be attributed to H_α and helium D3 combined.

For reference to the spectrum of the nebula, a photograph by Dr. Fath with a spectrograph of small dispersion was employed. In this spectrogram the lines λ5007 and 4958 are visible as a band, the hydrogen series from H_β to H_ζ and the line λ3727 separately, and a faint band in the blue corresponds to H_ϵ 4472 and 4390. Nu 4363 is merged into H_δ owing to the small dispersion.

The following is a list of the photographs obtained: the unscreened exposures were for comparative purposes:—

Plate No.	Date 1919.	Screen. Wratten No.	Plate.	Efficient Range of Wave-lengths.	Exposure.	Conditions.
1	Jan. 23	Unscreened	Imperial Flashlight	3200–4750	10′	Good
2	,,	8	Wellington Anti-screen	4700–5500	60	,,
3	,, 26	Unscreened	Imperial Flashlight	3200–4750	1	,,
4	,, ,,	,,	,,	3200–4750	4	,,
5	,, ,,	2	,,	3850–4750	60	,,
6	Feb. 18	49	,,	4300–4750	30	Misty
7	Mar. 3	2	,,	3850–4750	15	Good
8	,, 5	15	Wratten Panchromatic	5100 and longer	60	,,

Plate 2 (Nebulium Image).

A great disadvantage of screen No. 8 is that it transmits about 35 per cent. of H_β in addition to Nu 5007 and 4958, but the sensitivity curve of the isochromatic plate is in the direction of still further diminishing the effect of H_β, and it is evident from a comparison of Plate 2 with Plate 7 (hydrogen) that this radiation does not contribute appreciably to the image obtained.

One of Keeler's green screens was estimated to transmit about 50 per cent. of H_β*, and Hartmann's screen transmitted all three radiations "almost without loss." As Hartmann used a type of isochromatic plate which was specially sensitive to blue-green, his nebulium photograph (reproduced in *Ap. J.*, 21) cannot be regarded with much confidence, especially as it shows strong nebulosity round the star Bond 734, to the north of θ, while my photograph is in agreement with Keeler's in showing a well-defined star disc with only a trace of nebulosity just visible on the negative.

Keeler drew attention to this, and concluded that H_β was responsible for the faint nebulosity visible on his longer exposure. Another remarkable point of difference between Hartmann's photograph and my plate No. 2 is in the large negative colour index of the fainter stars, especially to the west of the nebula. Hartmann's photograph shows practically no difference in the size and number of the star images between his blue-green plate and the ultra-violet plate. His plate was certainly exposed for double the time of Plate 2, but after allowance is made for this, one is driven to the conclusion that the screen he used must have transmitted a much greater range of spectrum than was claimed, and the very extended image he obtained cannot be reconciled with either Keeler's results or mine. Unfortunately no reproductions of Keeler's photographs are available for direct comparison. Plate 2 gives a strong image of the Huyghenian region which compares very well in density with Plate 7. The nebulium image extends very little beyond the Huyghenian region, even such a conspicuous feature as the Proboscis Major being extremely faint.

A comparison with Plate 7 shows strikingly how largely hydrogen predominates in the Proboscis Major and the outer regions of the nebula.

Perhaps the best means of comparing the two photographs is to view them together in a stereoscope, and a direct comparison of the number and size of the star discs can also be made. It would be possible to reduce the transmisson of H_β to a negligible amount by using the Wratten screen No. 10, although the transmission of Nu 5007 and 4958 would also be weakened. Further work in this direction is desirable with longer exposures, as the presence of H_β introduces some uncertainty in all the results hitherto.

Plates 5 and 7 (Hydrogen Images).

Both these exposures gave successful results, and the image given by Plate 5 (60^m exposure) is about equal in density to an

* The other screen used by Keeler transmitted a still larger proportion.

unscreened exposure of 15^m. A comparison of this plate with Hartmann's ultra-violet plate shows no difference in detail but a good deal of difference in the comparative brightness of various features, and on the whole Hartmann's conclusions are confirmed. At the same time it is evident that the hydrogen image is of greater extent than he thought, and the bright feature about 10′ west of θ which he attributed to $\lambda 3727$ is faintly visible on Plate 5. A more detailed comparison with ultra-violet plates of graduated exposures would probably yield interesting results. An attempt was made to get a photograph of the $\lambda 3727$ radiation, but the ultra-violet screen employed did not transmit sufficiently, and only the brighter stars were shown.

Plates 6 (H_δ *and* H_ϵ) *and* 8 (H_α *and He* D_3).

Plate 6 was exposed under poor conditions, and only gave a faint image of the Huyghenian region, which showed no details differing from an unscreened plate of comparable density.

Plate 8, however, gave an image of the red radiation at H_α, confirming the presence of this line in the Huyghenian region, and especially of the "reddish fringe" mentioned by Barnard and Keeler along the south-eastern edge. A similar exposure has, I find, been made by Hale with the same result, but it was not published by him.

The following conclusions can be drawn from the screened photographs so far obtained:—

(1) The nebulium radiations at $\lambda\lambda 5007$ and 4958 are confined to the Huyghenian region and its neighbourhood. They are very faint in the Proboscis Major, and do not appear round the conspicuous star Bond 734, which with even a minute's unscreened exposure is surrounded by conspicuous nebulosity. Hartmann's photograph dealing with these radiations gives too great an extension, besides disagreeing in important details with Keeler's results and my own.

(2) The hydrogen radiations, with the exception of H_α, extend for a great distance, and there seems good reason to think that they are coextensive with $\lambda 3727$, although the latter is the brighter in various features of the outer regions.

The radiation H_α is present in the Huyghenian region, but it is uncertain if it extends to the outer regions. In any case, H_α is faint compared with the normal hydrogen series.

Harborne:
1919 *May* 21.

The Stellar Magnitude Scales of the Astrographic Catalogue. Fourteenth Note: The Tacubaya Magnitudes ($-16°$) *and the Cape Magnitudes* ($-43°$). By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. In the eleventh note (*M.N.*, 78, 54) the promise of another Tacubaya zone ($-16°$) was announced; and in the twelfth note (*M.N.*, 78, 589) it is stated that another Cape zone ($-43°$) was being counted. The Tacubaya zone was received from Sen. Gallo a few weeks since, the counts of the Cape zone were completed some months ago, and it is the principal object of this note to add this material to the steadily accumulating information about the southern hemisphere. But in view of the gatherings for scientific discussion among the Allies, and the prospects of a resumption of co-ordinated work on the Astrographic Catalogue, a few words may be added on the primary purpose of these investigations.

2. At the last meeting of the Comité Permanent in April 1909 it was resolved to initiate considerable investigations for the unification of (*a*) the places, (*b*) the magnitudes of the stars measured. I have grave doubts (which have been put forward on several occasions) as to the advisability of *either* project in the form in which it was left by the Committee. At the moment project (*a*) does not concern us. The main lines of project (*b*) may be thus described: Assuming that the magnitude scale is *uniform for each observatory*, but with errors special to each, let us investigate these special errors by photographs of sample regions on the same plate as the region of the Pole.

3. Now, the main purpose of the present series of papers is to test the fundamental assumption here made. If the scale at each observatory is *not* uniform, the work based on this assumption may be in large part wasted. My contention is that we can, with comparatively little labour (which is in no case wasted, since the work involved must in any case be carried out at some time), exhibit the consequences of this assumption, by simply counting the stars assigned by any observatory to the same magnitude, whatever that magnitude may be. If, on the one hand, it turns out that the scale *is* satisfactorily uniform, then these counts will in any case be useful to tell us how many stars of a given magnitude there are in that part of the sky. But, on the other hand, they may arrest scrutiny earlier and thus save waste, for they may show that the scales are not uniform enough to allow of systematic correction.

4. To take an illustration from another department of astronomy. It is advisable to determine the division errors of a circle, and although the work is serious—"six weeks' convict labour" it has been called by the late Professor Lewis Boss—the labour is well expended, since the division errors remain constant, surviving even the transportation of the instrument over long distances, as Boss himself proved by repeating the laborious investigation

after transportation. But suppose the division errors had *not* been stable, suppose that they were so unstable that observations of the same star were apt to differ by several seconds on different nights, it would then have been sheer waste of time and labour to investigate the errors unless the instability could be traced to its source and completely corrected. My assertion is that the division errors of the photometric scales of the various observatories are rendered unstable (by personality of the measurers and other less obvious causes) to an extent which renders the proposed investigation waste labour until the causes of instability have been first investigated and if possible removed.

5. As another piece of evidence in support of this assertion (many have already been given) let us consider the measures of zone -43° (given below) at the Cape Observatory under two broad classifications:—

(α) All the bright stars for which measures are recorded between 1·99 and 1·00. (There are stars brighter than these for which the measures are 2·00 and more, but we exclude these for the moment.)

(β) All those stars with *measures* less than 1·00. The lower limit is assigned by passing from measures to estimates. We are not now raising any question about this limit except that of its constancy throughout the zone -43°, nor making any assumption beyond that tacitly implied in the publication of such results by an observatory with the reputation of the Cape, viz. that a volume of results is intended to be on a uniform plan unless some special warning is issued with it.

6. Let us, then, compare the numbers of such stars from 0^h to 17^h and from 18^h to 23^h; we find them to be as follows:—

	(α).	(β).	$\log\alpha - \log\beta$.
0^h–17^h	11045	8466	+0·115
18^h–23^h	2978	4233	−0·154

The difference in $\log\alpha - \log\beta$ is 0·269, corresponding to more than half a magnitude. The consistency of the difference is shown in Table I. by giving $\log\alpha - \log\beta$ for each hour of R.A.

TABLE I.

Cape -43°: *log* α–*log* β *for each hour of R.A. in units of* 0·01.

h		h		h		h	
0	+11	6	+ 8	12	+ 6	18	−14
1	+10	7	+11	13	+17	19	−22
2	+22	8	+22	14	+26	20	−16
3	+17	9	+ 2	15	+17	21	−12
4	+24	10	+ 7	16	+17	22	− 8
5	+10	11	− 5	17	+17	23	−11

7. As far as 17^h there is but one negative value (at 11^h); after it there is no positive value. The probable error of an hourly difference is about ± 4 (in the adopted unit of 0·01), so that the abrupt change at 17^h–18^h is almost certainly real. A glance at the table in the introduction to the volume suggests that it is probably a case of personality, the plates after 17^h being chiefly measured by ES and NM, initials which scarcely occur at all before this point.

8. But it is undesirable to pursue the matter further here, since personal differences can only be studied to the best purpose at the observatory itself, where all the circumstances are fully recorded and understood. The point to which attention is here emphatically drawn is, that an elaborate scheme for testing individual plates in detail, such as that projected at the last meeting in 1909, though it may give useful information about those particular plates, is foredoomed to failure as an examination of the whole material, unless the homogeneity of that material is first ascertained and corrections obtained for discordances when found. That such discordances are the rule rather than the exception it has been the main purpose of these papers to demonstrate, though other investigations have arisen incidentally.

9. We may now give the main counts for Cape $-43°$; those for $-41°$ have been given already (*M.N.*, **75**, 59). As regards $-42°$, the harmonic analysis was given (*M.N.*, **78**, 589) but not the actual counts, which may as well now be given alongside those of $-43°$ in Table II. The heading **2** includes all stars from 2·00 to 2·99; it will be clear that they can easily be counted by concentrating attention on the integer. The heading [0] is limited in one direction by 0·99, but in the other by *about* 0·6, after which measures are replaced by estimates. Attention has already been called to the uncertainties of the lower limit.

10. In Table III. is given the harmonic analysis of the $-43°$ results in the same form as Table XI. on p. 589 of *M.N.*, **78**.

TABLE

TABLE II.

Counts of Stars in Cape Zones −42° and −43°.

	−42°					−43°				
R.A.	3	2	1	[0]	−	3	2	1	[0]	−
0h	4	28	186	172	410	14	30	197	153	531
1	12	26	242	278	377	14	34	182	144	577
2	9	23	253	156	637	8	25	186	113	565
3	5	28	293	253	621	4	29	280	192	849
4	16	51	382	413	849	9	33	286	167	1057
5	22	76	639	608	1402	10	52	529	417	1654
6	28	88	681	754	2105	29	77	664	547	2078
7	30	130	1032	874	2745	38	110	906	711	3553
8	46	155	1370	1166	3198	58	188	1736	1039	3819
9	12	68	1159	571	2174	19	103	1030	985	3038
10	17	60	1036	741	2625	27	72	659	559	2283
11	19	53	497	282	1961	24	66	536	606	1992
12	10	34	335	258	1252	6	22	313	277	1237
13	27	54	551	328	1807	16	44	570	386	2312
14	9	35	540	380	2192	14	51	649	357	2200
15	10	77	672	619	3440	19	85	764	518	2610
16	36	61	688	942	3367	15	58	517	346	2112
17	15	90	1169	1540	7066	15	110	1041	949	4570
18	19	116	1143	2331	8566	32	104	1193	1648	3736
19	10	60	645	580	3006	9	58	622	1019	1799
20	17	35	411	372	1478	16	55	434	638	1125
21	8	24	324	280	1132	9	38	319	420	641
22	9	26	252	271	488	8	33	210	249	336
23	11	22	187	156	486	11	37	200	259	447
Total	401	1420	14687	14325	53384	424	1514	14023	12699	45121

There are some conspicuous differences between the two zones, especially in the last columns. Compare these two columns, for instance, at 4^h and 18^h. At present we have no resource but to treat such differences as accidental.

TABLE III.

Harmonic Analysis of Residuals for Zone −43°.

d.	log No.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
3·00	9·98	+·054	+·006	−·100	−·031	+·037	+·131	+·101	+·018
2·00	0·71	+·031	+·056	−·105	−·082	+·033	+·073	+·051	+·009
1·00	1·67	+·012	+·026	−·082	−·107	+·006	+·005	+·043	+·007
[0·00]	1·94	−·034	+·063	−·142	−·095	+·008	−·025	+·044	+·030
All	2·38	−·007	+·015	−·057	−·107	+·027	−·025	+·053	+·031
Diff.	1·94	−·084	−·004	−·012	−·013	−·004	−·128	+·003	+·045
	1·00	−·043	−·002	−·006	−·007	−·002	−·066	+·002	+·023

The penultimate line is the excess of the sums of those for [0·00] and "All" over 2·00 and 1·00, and on dividing by 1·94 we get the final line.

For the first harmonic $R_0 = -·043$, $\theta_0 = 87°$.

[On p. 589, *loc. cit.*, the − sign of R_0 for zone $-42°$ has unfortunately been omitted in the text, though it is given correctly in the little table just below.] The accordance with other zones is undoubtedly affected by the systematic difference between the portions 18^h–23^h and 0^h–17^h, to which attention is called above; but the interference has affected the coefficient R_0 rather than the phase θ_0, which is in good accordance with those of $+38°$, $+41°$, and $+42°$, the four values being 91°, 77°, 106°, and 87°. These four zones give a good determination in this neighbourhood, as shown below; but before tabulating their means, let us first turn to the Tacubaya zone $-16°$, which also completes a set of four zones, two from Tacubaya and two from Hyderabad.

11. The counts, as kindly supplied by the Director of the Tacubaya Observatory, are given in Table IV. They were treated in the manner fully explained for zone $-15°$ in *M.N.*, 78, 62, etc., and the harmonic analysis is given in Table V.

TABLE IV.

Tacubaya Zone $-16°$.

Hour.	No. Plates.	Brt.	8·5.	9·0.	9·5.	10·0.	10·5.	11·0.	11·5.
0	8	49	41	59	208	279	272	146	119
1	7	53	37	45	113	195	150	233	78
2	8	53	38	54	128	220	202	123	216
3	7	46	41	57	116	149	96	221	81
4	8	63	48	71	166	131	137	204	193
5	7	69	71	96	236	299	326	561	331
6	8	70	61	138	327	441	682	1428	1176
7	7	68	66	176	332	602	2013	3422	2342
8	8	91	107	143	314	671	1041	1764	1813
9	7	59	44	122	74	267	654	598	718
10	8	51	53	80	136	232	412	450	416
11	7	36	41	38	69	152	140	439	412
12	8	31	33	73	128	204	342	447	263
13	7	32	34	68	98	104	340	342	155
14	8	32	38	57	140	159	237	585	646
15	7	32	26	61	115	258	240	323	261
16	8	39	27	35	108	255	441	484	159
17	7	50	41	177	232	464	1071	806	181
18	8	93	66	196	509	478	1716	2641	1809
19	7	44	89	172	289	594	1382	934	803
20	8	64	112	129	251	564	838	1237	981
21	7	33	43	78	173	268	410	621	543
22	8	45	51	84	127	225	313	280	316
23	7	50	30	65	121	221	215	197	231
Sums		1253	1238	2274	4510	7432	13670	18486	14243

TABLE V.

Tacubaya $-16°$.

Harmonic Analysis of the Residuals formed from Table IV.

Tac.	log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
8·0	0·51	+·105	+·113	+·006	+·065	+·037	−·015	+·006	−·008
9·0	1·13	+·047	+·105	−·073	+·061	+·002	−·059	+·038	−·012
10·0	1·71	+·016	+·143	−·073	+·062	−·004	−·004	+·055	+·005
11·0	2·15	−·023	+·042	−·149	+·017	−·008	+·013	+·101	·000
All	2·26	−·015	+·026	−·171	+·028	−·021	+·011	+·102	−·008
	2·77	−·190	−·150	−·253	−·081	−·068	+·098	+·159	+·012
	1·00	−·069	−·054	−·091	−·029	−·024	+·035	+·057	+·004

The penultimate row is obtained by subtracting the sum of the first two from that of the last two, and then dividing by 2·77 we get the final row.

For the first harmonic $R_0 = -·087_0$, $\theta_0 = 52°$. The values of θ_0 obtained for zones $-15°$, $-16°$, $-17°$, $-18°$ are thus 19°, 52°, 35°, 316°.

12. We may now collect the results from $-15°$ to $-43°$, which fall into three groups, of which means are taken. The terminal groups are compact, the middle group could be improved in this respect.

TABLE VI.

Collected Coefficients for the Harmonics, corresponding to change of 1·00 *in log* N_0.

Zone.	sin θ_0.	cos θ_0.	sin $2\theta_0$.	cos $2\theta_0$.	sin $3\theta_0$.	cos $3\theta_0$.	sin $4\theta_0$.	cos $4\theta_0$.
−15	−·053	−·149	−·139	−·105	+·020	+·092	+·142	−·027
−16	−·069	−·054	−·091	−·029	−·024	+·035	+·057	+·004
−17	−·074	−·104	−·042	−·001	+·096	−·076	+·046	−·053
−18	+·047	−·049	−·129	−·043	+·065	+·019	+·088	+·007
Mean	−·037	−·089	−·100	−·045	+·039	+·018	+·083	−·017
−25	−·024	+·068	−·014	+·027	−·007	−·017	+·022	−·042
−32	−·043	·000	−·008	−·025	−·011	−·013	+·013	−·032
−34	−·061	+·001	−·002	+·063	−·015	+·014	+·057	−·004
Mean	−·043	+·023	−·008	+·022	−·011	−·005	+·031	−·026
−38	−·127	+·003	+·048	−·037	−·038	−·015	+·053	+·010
−41	−·115	−·027	+·001	−·016	+·005	−·028	+·010	+·001
−42	−·088	+·026	+·001	−·030	+·036	−·006	−·022	+·012
−43	−·043	−·002	−·006	−·007	−·002	−·066	+·002	+·023
Mean	−·094	·000	+·011	−·022	·000	−·029	+·011	+·012

13. Subtracting the means as indicated from the constituents above them, and taking numerical means of the residuals in each harmonic (treating $\sin\theta$ and $\cos\theta$ together), we get for the mean residual $\pm\cdot028$, $\pm\cdot023$, $\pm\cdot027$, $\pm\cdot021$. These means include any real differences between the zones.

14. The large value $\pm\cdot028$ for the first harmonic is principally due to the large discordances of zone -18° and -43° in $\sin\theta_0$, and of zone -25° in $\cos\theta_0$. The change in $\cos\theta_0$ between -18° and -25° is marked. It seems scarcely possible to admit accidental errors which would reconcile the large and consistent negative value from -15° to -18° with the small positive value which persists from -25° to -43°. It is perhaps worth remarking that two well-separated observatories (Tacubaya and Hyderabad) are concerned in the former group, and three (Cordoba, Perth, and the Cape) in the latter.

15. Is it conceivable that the change in site from the northern hemisphere to the southern can be responsible for so large a systematic *differential* effect? For it must be remembered that the coefficients tabulated are the *differences* of log N for bright and faint stars. If this is really the cause, the information is almost as important as if the source lay in the stars themselves, for there must be some hitherto unsuspected instrumental error of a kind which is liable to affect all our work seriously.

16. This possibility being open, it seems desirable to test it by photographing the same zone at a northern and a southern observatory. Thus if zone -22°, which is being photographed at Santiago and measured at Oxford, were also photographed at Hyderabad, the comparison should go far to settle the question raised.

Note on a Curious Instance of Opposite Proper Motions.

By Ethel F. Bellamy.

(*Communicated by Professor H. H. Turner.*)

In forming ledgers of the plates in the Vatican zones of the *Astrographic Catalogue*, I came across a curious instance of two stars, Helsingfors A.G.C. 13277 and 13278, within two minutes of arc in the heavens, and with opposite proper motions. They occur on two plates taken twelve years apart, and with different centres, as below :—

Centre of Plate.		Date of Exposure.	H 13277.		H 13278.	
R.A.	Dec.		Δx.	Δy.	Δx.	Δy.
h m	°					
22 24	+60	1901·789	+·007	−·001	−·010	·000
22 32	+61	1913·846	+·012	−·001	−·017	+·001

The Δx and Δy are the differences, in units of réseau intervals, of the Vatican standard co-ordinates from the standard co-ordinates calculated from the Helsingfors places, which are dated 1874·9, and when reduced to 1900·0 are as follows:—

	Mag.	R.A. (1900).	Dec. (1900).
		h m s	° ′ ″
H 13277	9·0	22 31 14·70	60 16 53·4
H 13278	7·5	22 31 22·50	60 18 34·1

The proper motions are roughly $+9''$ and $-13''$ per century, and are confined to the R.A.'s. If they had been in the directions of the two apices of proper motions, reference to a globe shows that there should have been a sensible P.M. in Decl. for at least one star. The association of the two stars is not *very* close, although they have consecutive numbers in the Catalogue, but perhaps determinations of parallax would be interesting.

On the Light Curves of Long-Period Variables. By J. G. Hagen, S.J., For. Assoc., R.A.S.

1. The various discussions on the classification of long-period variable stars which have appeared in the *Monthly Notices* and in the Journal of the *Brit. Astr. Assoc.* suggest a comparison with a similar division proposed by the late Professor Pickering in *H.A.*, **57** (1907). Pickering says indeed (p. 209) that his division into five groups "must not be considered as establishing different classes of long-period variables, but simply as indicating the different forms of light curves." Following the same line of thought, I consider only the geometrical shape of the curves apart from any physical interpretation.

The five groups of Pickering are these: curves with broad maxima, with broad minima, with rapid increase, with rapid decrease, and curves of uniform variation. The two types of broad maxima and of rapid decrease contain only four curves each among a total of sixty-seven, and may be considered as irregular exceptions to the more general class of uniform variation, at least as long as more stars of the kind and more accurate light curves are not available. Thus readjusted, the three groups represent a well-divided classification of the light curves, and, what is more interesting, they seem to be in close relation to the classifications of Professor Turner and of the Rev. T. E. R. Phillips (*M.N.*, **78**, 92, and *J.B.A.A.*, **27** [1917] 6).

2. The comparison between the three classifications is much facilitated by the fact that the list of curves analysed by Phillips (including those rejected as doubtful or discordant) completely covers Pickering's groups, excepting the star V Cancri (of rapid

increase), thus yielding sixty-six curves in common. Furthermore, the identification of the stars is rendered very easy by the arrangement of the Harvard volume **57**. Plates i. and ii. give Pickering's numbers, and table xii. (p. 202) furnishes the names of the variables, with all the quantities needed for computation.*

For the sake of clearness it seems best to present first the final results in form of a table and then to explain the single columns in succession.

Classification of Sixty-six Light Curves.

Groups.	Number.	α.	Range of ϕ_3.	Mean of ϕ_3.	Range of Mag.
					m
I. Uniform variation	30	−·04	338°	187°	4·66
IIA. Broad minima	17	−·15	115	125	4·76
IIB. Rapid increase	19	−·21	86	97	5·88

The first and second columns contain the three groups of Pickering and the number of curves in each. The numbering I., IIA., IIB. forestalls the final conclusion, as will be seen later.

The α in the third column is the quantity used by Professor Turner to express the relation between rise and fall in the light curves:

$$\alpha = \{2(M - m) - P\}/P.$$

The values of $(M - m)/P$ are taken from *H.A.*, **57**, table xii., and were combined into means for each group. There are seven stars with positive α in Groups I. and IIA., which have disappeared in the means and which are provisionally treated as accidental differences. An inquiry into that special class of curves whose α are positive would require more accurate light curves, as Professor Turner has pointed out. It appears that the variation in the first group is not strictly uniform, the rise being, on the whole, shorter than the fall.

The angles ϕ_2 and ϕ_3 are copied from the two tables of Phillips, and designate, as is known, the phases of the second and third harmonics. The range of ϕ_2 means simply the interval between the extreme values in each group. In collecting the ϕ_3 into means, three of them had to be changed by 360° so as to make the residuals from the means less than 180°, as they should be.

The range of magnitude is likewise taken from the Harvard table xii.

3. The conclusions from the above table are somewhat impaired by the fact that the numbers of the different columns do not refer

* To those who wish to identify the stars it will be useful to have a few printing mistakes corrected:—

H.A., pp. 202–203, for 001919 read 001909.
Plate I., 1st column, ,, 122903 ,, 122803.
,, 3rd ,, ,, 093634 ,, 093934.

to identical light curves, though they belong to the same stars. Mr. Phillips mentions, besides the Harvard curves, those of the *B.A.A.*, of Rousdon and others, and often combines two or three curves into one mean. Now, it may well be that the quantities $(M - m)/P$ and the range of magnitude of the *H.A.* do not exactly correspond to the curves used by Phillips, and that some of these latter curves would have been differently grouped by Pickering. Yet the general feature of the result cannot thereby be essentially falsified.

A first result of the comparison is shown by the steady decrease of α from the first to the third group. This is not surprising when the shape of the curves is considered, but it may be of special significance in reference to the following columns.

The second and main result can briefly be summarised in the statement that *Group I.* of Phillips is identical with the first group of Pickering, and that *Group II.* is split into sub-groups corresponding to the second and third of Pickering. In fact, the two classifications are made up from the same stars, with few exceptions. Among the curves of "uniform variation" there are seven which Phillips reckons in his *Group II.*, yet three of them are near the place where the two lines in *Diagram I.* meet and where the separation may be doubtful. Again, the group of "broad minima" has one star among seventeen, and that of "rapid increase" two among nineteen, which fall into Phillips' *Group I.*; but one of these three is also near the juncture of the two straight lines. Thus we have only six certainly discordant stars among the sixty-six common to both classifications, a number almost insignificant in view of the fact that the light curves used in both cases are not identical.

The sub-groups IIa. and IIb. can be neatly traced on Phillips' *Diagram I.* when the points representing the stars of "broad minima" are specially marked. They all fall into the upper half of the slant line, while those of "rapid increase" lie below.

The range of ϕ_2 in the three groups becomes gradually narrower as ϕ_3 decreases, which confirms the subdivision.

In the last column the interesting, though not surprising, fact is revealed that the downward increase of range in magnitude which was found by Phillips is not due to the whole of his *Group II.* but to sub-group IIb., *i.e.* to the curves of "rapid increase."

It is hoped that this comparison between three apparently different classifications of the light curves may bring those of Professor Turner and of the Rev. T. E. R. Phillips into closer connection, and that a geometrical interpretation may be opened for the peculiar behaviour of the phase-angles in the harmonic analysis.

Vatican Observatory:
1919 *March* 25.

Star-Discs. By A. E. Conrady.

Eighty-five years ago Airy * determined the form of the image of a point of light in the focal plane of a perfect optical instrument which results from the interference of the light meeting in that plane.

The Airy spurious disc naturally represents an unattainable ideal, inasmuch as both perfect correction of the instrument and exact focussing are assumed. By far the most valuable extension of Airy's work is due to Lord Rayleigh,† who took spherical aberration, coma, and astigmatism into consideration. The well-known rule that the waves converging towards an image-point may be allowed to depart from the ideal spherical form to the extent of one-quarter wave-length without serious loss of efficiency was the chief outcome of this work.

Lord Rayleigh did not determine completely the light distribution in the image of a point in the presence of spherical aberration; he showed that with a quarter wave-length of wave-distortion the central intensity was only diminished by about 20 per cent., which he rightly considered an admissible amount.

A more detailed knowledge of the light distribution at and near the focus of an optical instrument is highly desirable; in fact the adequate explanation of the results obtained by large instruments absolutely depends on this knowledge being secured. Having failed to induce others to take up this work I have undertaken it myself, and the results have already proved an abundant reward for the labour involved.

As practically all problems of this type lead to integrals which can only be solved analytically by development into series, which frequently are of a highly inconvenient form for numerical evaluation, I decided to work entirely by "mechanical" quadratures of the simplest type. High precision of the numerical values would be absolutely futile, as every minute change of focus or of aberration appreciably alters the result; consequently it was decided to aim at an accuracy to 1 part in 1000 for the maximum intensities, and to about one ten-thousandth part of the maximum intensity for the feebler luminosity in the outer part of the central disc and in the surrounding rings. This brought the work within the capacity of available tables, such as Barlow's table of squares, etc., and especially Crelle's multiplication table. In this way the work was so far reduced that a few hours now suffice to determine the light distribution up to the outer boundary of the second diffraction ring for any selected case of either primary or higher spherical aberration.

Only the usual *circular* aperture of a centred optical system was taken into consideration. Such an aperture may be considered

* *Camb. Phil. Trans.*, 1834.
† *Scientific Papers*, vol. i. pp. 415–453.

as built up of an infinite number of concentric zones; the integration may accordingly be carried out first for such a narrow circular zone, and may then be extended to cover the whole aperture by forming the sum of the contributions of all the zones from the centre to the margin of the clear aperture.

I. Integration for an infinitely narrow concentric zone. All points of a circle, such as OPP′ shown in perspective in fig. 1, necessarily are equidistant from *any* point F on the line OF erected at right angles to the plane of the circle in its centre O. It follows that an infinitely narrow circular zone of a light-wave emerging from an optical system has no definite focus: it produces maximum intensity at any point on the centre line OF. If we consider some point such as F′ not on the centre line, then the light from different parts of the circular zone will reach *this* point with a difference of phase which is proportional to sin E if the angle E is reckoned from the radius OR at right angles to the plane defined by O, F, and F′. If the vibration reaching F′ from R is expressed simply by $a \cdot \sin x$, where $x(=2\pi \cdot v \cdot t/\lambda)$ is an angle increasing by 2π for every complete vibration, then the vibration reaching F′ simultaneously from a point P will be expressed by $a \sin(x + W \cdot \sin E)$, W being the maximum difference of phase reached at Q in the plane OFF′. To every point P we can find a corresponding point P′ at the same angle E, but counted in the opposite direction, for which the vibration is therefore expressed by $a \sin(x - W \cdot \sin E)$. If we combine such *pairs* of points by addition, we obtain the expression for the combined vibration $a \sin x \cos(W \cdot \sin E)$, the terms in $\cos x$ cancelling each other.

The integration of this expression from E = 0 to E = 90° gives half the total effect produced at F′, for the lower half of our circle would obviously reproduce the same results. But as we require only relative values, referred most conveniently to the maximum amplitude produced at F, we calculate the amplitude at F′ as

$$A = a \int_0^{90°} \cos(W \cdot \sin E) dE$$

and select the constant and arbitrary factor "a," so that the value A becomes 1·000 for W = 0. The value of W evidently grows in direct proportion with the angular distance FOF′ of the point F′ considered and with the radius r of the circular zone. For the purpose of tabulation we assume a series of convenient values (W = 0, 40°, 80°, etc., to 600° were used in my direct calculations) and then complete the work by interpolation. The integration was carried out by the formula which results from integrating Stirling's interpolation-formula, by calculating W . sin E for E = $2\frac{1}{2}$°, $7\frac{1}{2}$°, etc., to $87\frac{1}{2}$°, *i.e.* for the middle points of 18 sections of 5°, taking out the corresponding cosines and adding these up. Owing to the fact that these cosines pass through a maximum = 1 for E = 0 and become constant for E = 90°, because sin E is there constant, the correction by differences was zero at both limits of the

integral, and the simple sum therefore gave the correct value of the integral. The values so found are given in Table I.*

TABLE I.

Amplitude of Vibration produced by a narrow plane Circular Zone of a Light-wave at Points where the extreme difference of Phase is equal to 2 W Degrees, the amplitude in the axis of the Zone being taken as Unity.

W.	0.	100°.	200°.	300°.	400°.	500°.
0°	+1·000	+·372	−·379	−·098	+·300	−·020
10°	·992	·271	−·397	−·038	·297	−·066
20°	·970	·170	−·403	+·022	·285	−·109
30°	·933	+·072	−·396	·079	·265	−·148
40°	·882	−·020	−·378	·132	·238	−·182
50°	·818	−·105	−·350	·179	·204	−·210
60°	·744	−·182	−·312	·220	·164	−·230
70°	·660	−·248	−·266	·253	·121	−·243
80°	·569	−·304	−·214	·278	·075	−·249
90°	·472	−·348	−·157	·294	+·028	−·248
100°	+·372	−·379	−·098	+·300	−·020	−·239

II. The integration over a given aperture consists in the summation of the effects of concentric zones at regular intervals. The light passing through such a zone of radius r is proportional to $2r\pi dr = \pi d(r^2)$. This points to the adoption of r^2 as the integration variable instead of r itself, for by doing so all the zones will contribute an equal share of the total light. Preliminary trials suggested a division of the whole aperture into 16 parts as amply sufficient for the degree of precision aimed at, and infinitesimal zones were accordingly selected at the centre and margin, and at 15 intermediate positions corresponding to successive $\frac{1}{16}$ths of the total area. Taking the full semi-aperture as unity, the radii of the successive selected zones therefore are 0, $\sqrt{\frac{1}{16}}$, $\sqrt{\frac{2}{16}}$ . . . $\sqrt{\frac{15}{16}}$, and 1. For the extreme marginal zone the originally computed values of A for E = 0, E = 40°, etc., to E = 600° were adopted. These correspond to a succession of points F′ (fig. 1) at equal intervals from the point F (where the intensity of the light is at its maximum) marking the centre of the spurious disc. At the first of these points the extreme marginal ray from one side of the aperture meets the corresponding ray from the opposite side with a difference of phase of $2 \times 40° = 80°$, and at the last point F this difference of phase rises to 1200° and corre-

* The integration by division of the angle E into 18 sections was proved to be abundantly sufficient by extracting from the original numbers those corresponding to division into 6 sections only (*i.e.* for E = 7½°, 22½°, etc.) and adding these. They give precisely one-third of the original values in every case.

sponds to the beginning of the third ring in the ideal Airy star-image. The points are close enough to give a satisfactory account of the whole light distribution up to this point.

The light from the inner zones will arrive at the same points with differences of phase less than those of the marginal zone in

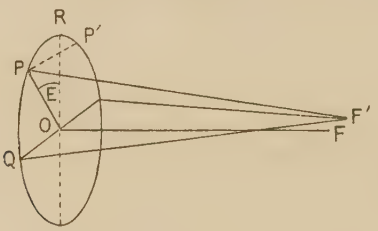


FIG. 1.

direct proportion to the respective radii of the zones. These phase-angles were therefore computed as $40^\circ \sqrt{\frac{1}{16}}$, $40^\circ \sqrt{\frac{2}{16}}$, etc., for the first point F′, and similarly for all the other points, and values of A for all the zones and points F′ could then be interpolated from Table I. The simple numerical integration of these values for each point F′ then gives the amplitude at that point in accordance with Airy's assumption of perfect correction of all aberration, and thus supplies a check of the whole work to this point. But for this integration a simple addition of the 17 values is entirely insufficient, and even Simpson's rule is hardly equal to it in the outer rings where the contributions of the successive zones vary violently in magnitude, and even change sign several times. I therefore adopted the extension of Simpson's rule, which consists in taking 5 successive values of the function to be integrated (instead of 3 for Simpson's rule) and integrating the curve of the fourth degree which can be laid through these 5 points. The formula is:

$$\int = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4),$$

y_0, y_1, etc., being the calculated values of the function (in our case the A values for any given point F′), and h the regular interval of the variable (in our case $\frac{1}{16}r^2$) for which the y have been calculated.

Our 16 intervals call for 4 successive sections to which the formula has to be applied. As the resulting coefficients will remain unchanged in all the applications, they were applied once for all to the interpolated A values of each zone, and Table II. thus resulted. An additional factor 10 has been applied throughout in the construction of this table in order that the maximum amplitude in the centre of the ideal disc should come to 10 exactly. The simple addition of the vertical columns of Table II. gives the amplitudes at the successive points F′: these have been compared by interpolation from Lommel's table—an extension of Airy's—

Table II.

Values of Amplitude contributed by narrow Circular Zones enclosing successive sixteenths of the full aperture, the factors in 2nd vertical column having been applied in order to obtain the correct integral as the simple sum of the terms in the vertical columns.

W of marginal zone.			0°.	40°.	80°.	120°.	160°.	200°.	240°.	280°.	320°.	360°.	400°.	440°.	480°.	520°.	560°.	600°.
Zone o	Factor	$\frac{7}{8\sqrt{6}}$	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194	·194
$r^2=\frac{1}{16}$		$\frac{8}{9}$	·889	·882	·862	·829	·784	·727	·661	·587	·506	·420	·331	·241	·151	·064	−·018	−·093
$\frac{2}{16}$		$\frac{1}{3}$	·333	·329	·313	·289	·257	·218	·174	·127	·080	·033	−·011	−·050	−·082	−·107	−·125	−·133
$\frac{3}{16}$		$\frac{8}{9}$	·889	·868	·810	·715	·592	·450	·295	·140	−·006	−·134	−·237	−·311	−·350	−·356	−·331	−·278
$\frac{4}{16}$		$\frac{7}{8\sqrt{6}}$	·389	·377	·343	·289	·221	·145	·066	−·008	−·071	−·118	−·147	−·157	−·147	−·121	−·083	−·038
$\frac{5}{16}$		$\frac{8}{9}$	·889	·856	·758	·610	·424	·225	·029	−·139	−·265	−·340	−·357	−·322	−·244	−·136	−·018	·096
$\frac{6}{16}$		$\frac{1}{3}$	·333	·318	·275	·210	·131	·049	−·027	−·086	−·123	−·134	−·122	−·090	−·045	−·004	·049	·082
$\frac{7}{16}$		$\frac{8}{9}$	·889	·813	·708	·511	·278	·044	−·154	−·292	−·355	−·340	−·260	−·134	·006	·136	·226	·266
$\frac{8}{16}$		$\frac{7}{8\sqrt{6}}$	·389	·366	·300	·203	·093	−·012	−·096	−·145	−·156	−·130	−·077	−·012	·050	·096	·116	·108
$\frac{9}{16}$		$\frac{8}{9}$	·889	·829	·661	·420	·151	−·093	−·270	−·353	−·336	−·236	−·087	·070	·196	·261	·253	·181
$\frac{10}{16}$		$\frac{1}{3}$	·333	·309	·239	·141	·035	−·056	−·116	−·134	−·113	−·063	·000	·056	·092	·099	·076	·034
$\frac{11}{16}$		$\frac{8}{9}$	·889	·816	·615	·335	·042	−·197	−·335	−·349	−·256	−·095	·078	·211	·267	·233	·130	−·007
$\frac{12}{16}$		$\frac{7}{8\sqrt{6}}$	·389	·354	·259	·129	−·003	−·104	−·153	−·145	−·089	−·010	·063	·109	·113	·079	·020	−·042
$\frac{13}{16}$		$\frac{8}{9}$	·889	·804	·570	·258	−·051	−·271	−·357	−·304	−·148	·043	·197	·266	·229	·113	−·037	−·164
$\frac{14}{16}$		$\frac{1}{3}$	·333	·299	·206	·083	−·034	−·112	−·134	−·101	−·034	·038	·088	·099	·069	·015	−·041	−·077
$\frac{15}{16}$		$\frac{8}{9}$	·889	·790	·527	·185	−·128	−·321	−·349	−·232	−·035	·154	·258	·244	·128	·032	−·167	−·222
1		$\frac{7}{3\sqrt{6}}$	·194	·171	·111	·033	−·035	−·074	−·073	−·042	·004	·043	·058	·046	·015	−·021	−·045	−·046
Sum = amplitudes in Airy spurious disc			9·999	9·405	7·751	5·434	2·951	·812	−·645	−1·282	−1·203	−·675	−·031	·460	·642	·521	·199	−·139
$\frac{1}{16}$ sum squared = intensity of light			1·0000	·8845	·6008	·2953	·0871	·0066	−·0042	·0164	·0145	·0046	·0000	·0021	·0041	·0027	·0004	·0002
									First ring.					Second ring.				

and found to agree in every case within a few units of the last retained figure, which is of course uncertain to that extent owing to the accumulated error of the 17 items composing it. One-tenth of these amplitude values, squared, gives the corresponding intensity of the light in the successive points F′, and these intensities are also given as the figures which are really the most interesting.

As an additional test of the sufficiency of the adopted method of integration the summation was carried out by the same method, but using only 7 intermediate zones; the results throughout agreed as closely as could be expected. But when the number of zones was reduced to 4, disagreement began in the second ring and rapidly became worse. This may be taken as satisfactory evidence that 16 zones are ample for the purpose of this paper, and would probably allow of including the whole of the third ring in the work.

My chief object was to determine the form of the star-image when Airy's assumptions are not fulfilled, that is, when the waves converging towards the focus are either distorted by aberration or are received at a point within or beyond their axial centre of curvature. In all these cases the light from the various zones arrives at the adopted focal plane with a difference of phase which diminishes the intensity of the light in the centre of the disc and modifies that in the remainder. Any such difference of phase is most easily dealt with by resolving the vibrations into two components with a difference of phase of 90° and integrating separately for these. Again adopting the equation $a \sin x$ to express the vibration, a zone out of phase to the extent of an angle ϕ will correspond to the equation $a \sin (x + \phi)$, and this may be resolved into $a \cos \phi \sin x + a \sin \phi \cos x$, which represents two waves with a difference of phase of 90°. The two integrals represent $A \cos \psi$ and $A \sin \psi$ respectively, A being the amplitude and ψ the phase-angle of the vibration in the corresponding point F′, and the resultant intensity is therefore found as $(A \cos \psi)^2 + (A \sin \psi)^2 = A^2$.

In the case of an axial image-point and also in that of an extra-axial point, provided coma and astigmatism are absent, the difference of phase for a zone of radius r can always be expressed in the form

$$\phi = c_1 r^2 + c_2 r^4 + c_3 r^6, \text{ etc.},$$

which follows directly from the complete symmetry of the image with reference to the optical axis or principal ray. The first term represents simply curvature, and is therefore absent if the adopted focus is that of the rays close to the optical axis or principal ray. The second term—in r^4—is the ordinary or primary spherical aberration, and the following terms represent spherical aberration of higher order.

The first case to call for attention, and in many respects one of the most important, is that when the star-image is simply out of focus; the difference of phase is then of the form $\phi = c_1 r^2$, and as r^2 is our integration-variable, the difference of phase of successive zones is represented by successive $\frac{1}{16}$ of the marginal value of ϕ. In accordance with Lord Rayleigh's rule, the case of a quarter-wave

Table III.

Light Distribution when Star is out of focus or when aberration is present.

W. of marginal zone.		0.	40°.	80°.	120°.	160°.	200°.	240°.	280°.	320°.	360°.	400°.	440°.	480°.	520°.	560°.	600°.
Marginal zone $\frac{1}{2}\lambda$ out of focus	$(A\cos\psi)^2$	·4051	·3708	·2813	·1732	·0814	·0253	·0030	·0003	·0028	·0036	·0022	·0006	·0000	·0000	·0001	
	$(A\sin\psi)^2$	·4051	·3467	·2077	·0745	·0070	·0049	·0253	·0312	·0184	·0040	·0001	·0035	·0055	·0031	·0004	
	Intensity	·8102	·7175	·4890	·2477	·0884	·0302	·0283	·0315	·0212	·0076	·0023	·0041	·0055	·0031	·0005	
Marginal zone $\frac{1}{2}\lambda$ out of focus	$(A\cos\psi)^2$	0	·0006	·0076	·0280	·0556	·0717	·0624	·0347	·0089	·0000	·0051	·0110	·0095	·0036	·0001	·0010
	$(A\sin\psi)^2$	·4051	·3584	·2422	·1157	·0297	·0004	·0084	·0227	·0247	·0153	·0049	·0002	·0007	·0022	·0020	·0008
	Intensity	·4051	·3590	·2498	·1437	·0853	·0721	·0708	·0574	·0336	·0153	·0100	·0112	·0102	·0058	·0021	·0018
$\frac{1}{4}\lambda$ spherical aberration paraxial focus	$(A\cos\psi)^2$	·6082	·5504	·4031	·2309	·0946	·0211	·0003	·0036	·0079	·0062	·0021	·0001	·0005	·0012	·0011	
	$(A\sin\psi)^2$	·1919	·1604	·0882	·0247	·0003	·0084	·0208	·0192	·0079	·0004	·0017	·0057	·0057	·0023	·0001	
	Intensity	·8001	·7108	·4913	·2556	·0949	·0295	·0211	·0228	·0158	·0066	·0038	·0058	·0062	·0035	·0012	
$\frac{1}{4}\lambda$ spherical aberration marginal focus	$(A\cos\psi)^2$	·1920	·1796	·1462	·1028	·0609	·0292	·0102	·0020	·0000	·0008	·0013	·0006	·0003	·0001	·0000	
	$(A\sin\psi)^2$	·6081	·5256	·3286	·1313	·0197	·0020	·0259	·0390	·0276	·0090	·0003	·0018	·0043	·0030	·0006	
	Intensity	·8001	·7052	·4748	·2341	·0806	·0312	·0361	·0410	·0276	·0098	·0016	·0024	·0046	·0031	·0006	
1 λ spherical aberration midway between parax. and marg. focus	$(A\cos\psi)^2$	·1922	·1699	·1164	·0596	·0214	·0048	·0007	·0003	·0012	·0031	·0150	·0035	·0010	·0000	·0019	
	$(A\sin\psi)^2$	·6103	·5383	·3645	·1756	·0471	·0013	·0089	·0258	·0275	·0158	·0043	·0001	·0009	·0018	·0010	
	Intensity	·8025	·7082	·4809	·2352	·0685	·0061	·0096	·0261	·0287	·0189	·0193	·0036	·0019	·0018	·0029	
1 λ spherical aberration $\phi = r^4 - \frac{19}{12}r^2$	$(A\cos\psi)^2$	·2736	·2292	·1264	·0340	·0000	·0213	·0571									
	$(A\sin\psi)^2$	·0016	·0004	·0015	·0124	·0176	·0448	·0412									
	Intensity	·2752	·2296	·1279	·0464	·0176	·0661	·0983									
4 λ r^6 aberration giving $\frac{1}{4}\lambda$ at best focus	$(A\cos\psi)^2$	·3668	·3173	·1999	·0823	·0144	·0002	·0088	·0117	·0047	·0000	·0051	·0139	·0168	·0112	·0035	
	$(A\sin\psi)^2$	·3647	·3297	·2406	·1375	·0566	·0132	·0003	·0019	·0052	·0059	·0046	·0031	·0019	·0012	·0007	
	Intensity	·7315	·6470	·4405	·2198	·0710	·0134	·0091	·0136	·0099	·0059	·0097	·0170	·0187	·0124	·0042	
Ideal Airy disc	Intensity	1·0000	·8845	·6008	·2953	·0871	·0066	·0042	·0164	·0145	·0046	·0000	·0021	·0041	·0027	·0004	·0002

difference of phase for the marginal zone was first submitted to calculation. This gives the phase-angle of $5\frac{5}{8}°$ for the first zone, and successive multiples of this for subsequent zones. Two tables have to be prepared, one in which the horizontal columns of Table II. are multiplied by the cosines of the phase-angles of the zone to which they refer, another in which the sines of the same angles are used as factors. The vertical columns are then added up in both tables, the results squared, and the two squares corresponding to the same marginal value of W or to the same point F′ of the image added together. The results are given in the first section of Table III. They show that the intensity in the centre of the imperfectly focussed star-image is ·81 of that in the sharply focussed image, and that this percentage of loss is very nearly maintained throughout the brighter part of the central disc. But whilst the ideal star-disc sinks rapidly to zero-intensity before light appears in the rings, the out-of-focus image possesses a *penumbra* which fills this gap with light of about 3 per cent. of the maximum intensity and which nowhere goes down to zero. The light lost in the central part thus reappears in the penumbra, and must obviously diminish contrasts in the image of an extended object such as a lunar landscape. But it is very remarkable that the diameter and *relative* distribution of intensity of the central disc is almost absolutely identical with that in the ideal Airy disc; and as this is the only part visible in the case of fairly faint stars, we are justified in concluding that this amount of disturbance of the focus has no effect at all on the resolving power for double stars nor for planetary detail, provided that the contrasts are sufficiently great.

The calculation was repeated for the case when the focus is defective to the extent of a marginal difference of phase of 180° (twice the Rayleigh limit). The results are given in the second section of Table III. The central intensity now sinks to ·405 of the maximum possible; but there is still no decided change in the fall of intensity in the brighter part of the central disc, and the resolving power for faint stars or strongly marked detail is therefore still well maintained. The penumbra, however, now begins to become decidedly conspicuous, for its intensity represents a considerable fraction of that in the centre of the disc. There will thus be a decided halo even around fairly feeble stars with this departure from the true focus.*

* The linear change of focus df, equivalent to a given change dp in the difference of lengths of optical paths with which the marginal ray meets the axial one, is easily deduced. If the marginal ray meets the axial one under an angle U, and df is very small compared to the focal length, then at the extended focus the path of the axial ray will be increased by df, that of the marginal ray by the projection of df upon the marginal ray, or by $df\cos U$. Therefore the marginal ray acquires a lead $dp = df(1-\cos U)$. In a telescope of aperture A and focal length f we have with ample accuracy $U = A/2f$, therefore

$$dp = \tfrac{1}{8}df\left(\frac{A}{f}\right)^2, \quad \text{or} \quad df = 8dp\left(\frac{f}{A}\right)^2.$$

This applies to df and dp expressed in the same unit of length. Taking df,

It is very curious that this case does not seem to have been investigated before, for it has a most important bearing on the theory of the achromatic telescope. As is well known, the secondary spectrum renders it quite impossible to bring all the colours to the same focus. The best that can be done consists in establishing a minimum focal distance for the brightest rays (visually near $\lambda = 5550$). The other colours are then focussed in pairs at greater distances, which increase roughly with the square of their distance from the brightest rays in the glass spectrum. With all the ordinary crown and flint glasses C and F light is thus focussed in visually corrected instruments at about 1/2000 of the focal length beyond the focus of the brightest rays, with the obvious result that, whatever focus we may choose, most of the light will be more or less out of focus, and will produce spurious discs of the modified form which the above calculations disclose. It is evident that from this point of view it is of fundamental importance that the diameter of the central condensation of light is practically unaffected by a moderate change of focus; it explains at once the possibility of a highly defined image in the presence of a considerable secondary spectrum. Visually the light beyond the C and F lines is very feeble, and we may therefore assume that there will be no perceptible loss of definition if we so proportion our instruments that none of the light between C and F reaches the adopted compromise focus with more than $\frac{1}{2}$ wave-length of variation of phase. It is easily shown that with the above-stated magnitude of the secondary spectrum (1/2000 of the focal length for C and F) this will be the case if clear aperture A and focal length F, both in inches, are determined by the relation

$$F = 2{\cdot}8A^2.$$

This gives focal lengths below those usually adopted for object-glasses up to 5 inches aperture, but it calls for focal lengths in excess of those actually in use for apertures over 6 inches. Thus a 10-inch should have 23 feet focal length, and a 20-inch about 93 feet. We have here the explanation of the loss of definition, especially on extended objects, with these large instruments. What saves them from being useless is the parabolic law connect-

A, and f in inches, adopting $\frac{1}{48000}$ inch as a fair estimate of an average wave-length, and using n as a symbol for the number of wave-lengths corresponding to dp, we find

$$df = \frac{n}{6000}\left(\frac{f}{A}\right)^2.$$

For the ratio usual in refractors, $f : A = 15 : 1$, this gives

for $n = \frac{1}{4}$ $\quad df = {\cdot}009''$

for $n = \frac{1}{2}$ $\quad df = {\cdot}019''$.

But as we may go out of focus in either direction, these figures require doubling in order to represent the permissible range of focal adjustment corresponding to the Rayleigh limit and to twice that limit respectively.

ing the change of focus with the distance of any spectrum line from that for which minimum focal length is established, for in the case of light from the ends of the section midway between C and F and of half their separation the differences of phase will be only one-quarter of those for C and F, and they will be sufficiently well focussed if the focal length is about one-quarter of that given by the formula; nevertheless there will be a large amount of light entirely lost to the defined image, and rendered positively harmful by being spread out into a halo which obscures detail of low contrast.

Spherical Aberration.

Simple or primary spherical aberration gives a difference of phase at the paraxial focus expressed by

$$\phi = c_2 r^4.$$

Taking Lord Rayleigh's quarter-wave amount first, we have to multiply the A values of Table II. by cos and sin of $90^\circ \left(\frac{n}{16}\right)^2$, n being the number of the zone, and we must then add up and form the sum of the squares for each point in the diffraction image. The results are given in Table III. In almost exact agreement with Lord Rayleigh we find the central intensity as ·80 of the ideal maximum. Comparison with corresponding intensities in the ideal image again shows that throughout the brighter part of the central disc the same percentage of loss is maintained, and therefore that definition of the image is hardly affected. The light lost from the central disc is largely transferred to a penumbra reaching out well into the second ring, and this will lead to some loss of contrast in extended objects.

We must, however, note that it is quite arbitrary to select the paraxial focus as the one to be considered. Assuming spherical under-correction and a converging pencil, the focus will become progressively shorter for zones approaching the margin of the aperture. Geometrical treatment of the problem leads to a circle of least confusion coinciding with the focus of the rays from $\frac{3}{4}$ of the full aperture, and this would therefore have a claim to be studied as more likely to yield a good concentration of light. We will, however, go to the other extreme and determine the light distribution resulting at the focus of the marginal rays. Referring to fig. 2, we have the real wave, distorted by spherical aberration, represented by the curve PP′. If F_p represents the paraxial focus, a circle PP_p with PF_p as radius will represent the ideal wave for the paraxial focus; and if F_m is the point where the marginal normal of PP′ cuts the axis, this will mark the marginal focus, and a circle PP_m with PF_m as radius will represent the ideal wave form for the marginal focus. The separation between PP′ and PP_p represents the spherical aberration $\phi = c_2 r^4$, which we have already studied. The separation between PP′ and PP_m is the corresponding aberration for the marginal focus. We

can determine this by noting that the marginal ray is a normal of both PP′ and PP_m, and therefore that the marginal elements of these curves stand at the same angle with reference to PP_p. The angle between PP′ and Pp we can determine at once; it is obviously given by the differential coefficient

$$\frac{d\phi}{dr} = 4c_2r^3.$$

PP_m will have a greater curvature than PP_p, say by c_1r^2; the angle between corresponding elements of PP_m and PP_p is there-

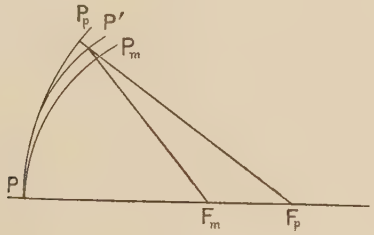


Fig. 2.

fore given by the differential coefficient of c_1r^2, and comes out as $2c_1r$. The condition that the two angles shall be equal for the marginal ray, for which we take $r = 1$, therefore leads to the equation

$$4c_2 = 2c_1 \quad \text{or} \quad c_1 = 2c_2.$$

The aberration with reference to the marginal focus, therefore, is represented by

$$\phi_m = c_2r^4 - 2c_2r^2.$$

Again, taking $c_2 = \frac{1}{4}$ wave-length $= 90°$, we can calculate ϕ_m for our 16 zones and again tabulate the sine and cosine terms as before. The results are given in Table III. They show that the central intensity at the marginal focus is precisely the same as that at the paraxial focus. The intensity, however, falls off more rapidly as we proceed outward, and the central disc is therefore slightly *smaller* and more favourable to high resolving power than that at the paraxial focus. The penumbra, however, is brighter, and therefore more objectionable, at the marginal focus.

There is, however, a further conclusion to be drawn which reduces the interest in both these foci almost to zero. By a suitable change of focus we can give to the coefficient of r^2 in our last equation any value we please; we can therefore inquire what value will reduce the residual differences of phase to its absolute minimum. Taking our range of r from 0 to 1, this simple problem leads to the equation

$$\phi \text{ (minimum)} = c_2r^4 - c_2r^2,$$

which gives zero value for $r = 0$ and for $r = 1$, and a maximum for $r = \sqrt{\frac{1}{2}}$ which amounts to

$$\text{maximum } \phi = \tfrac{1}{4}c_2,$$

or only $\frac{1}{16}$ of a wave-length in our case. It is not worth while to carry out the integration for this case; it would lead to a central intensity and a light distribution to all intents and purposes identical with that in the ideal Airy disc. This identity of result in fact goes further; for a comparison of the tables for the paraxial and marginal focus, which corresponds to a change of focus equal to twice that adopted in our first study, show practical identity with the result of a similar change of focus in the absence of spherical aberration. One-quarter wave-length of primary spherical aberration is therefore utterly harmless, and incapable of detection by direct observation at and near the sharpest focus. Incidentally our equations for paraxial, marginal, and best focus show that the latter lies precisely midway between the others, not at $\frac{3}{4}$ of their distance, as the geometrical treatment suggests.

It is curious that the enormous reduction of the effect of a given wave-distortion due to spherical aberration by a suitable change of focus has escaped previous investigators, although they realised that there would be some effect of that kind.

We feel impelled to try next the effect of a difference of phase of one whole wave-length at the *best* focus, that is, when c_2 is put $= 360°$ in the formula

$$\phi \text{ (minimum)} = c_2r^4 - c_2r^2.$$

When these integrations are carried out we obtain the results tabulated in Table III. The central intensity comes out at ·80, about the same value as in other cases of a quarter-wave distortion, and the fall of intensity also agrees closely with the previous cases. We may therefore accept one whole wave-length of primary spherical aberration as the amount beyond which serious deterioration sets in.

With this amount of spherical aberration we can secure another remarkable result, which points the way to the explanation of very high double-star resolutions such as are from time to time recorded by reliable observers.

It should first be stated that Dawes' classical figure for the diameter of observed star-discs corresponds to about 90° in the scale of marginal phase-difference W, under which our intensities are tabulated. The Airy disc has about $\frac{1}{2}$ of the central intensity at that point. So high a theoretical intensity at the measured *edge* of the disc in itself suggests that it must have been an abnormal image. We shall now see that spherical aberration can give such an image, although at the expense of great loss of intensity in the centre and of a very decided halo.

The finding of the best result of this type calls for many trials, and these have not yet been carried out; but if we calculate the

light distribution corresponding to the formula—realisable by a mere change of focus—

$$\phi = c_2 r^4 - \tfrac{1}{2}\tfrac{9}{2} c_2 r^2,$$

which leads to a focus a little nearer the marginal focus than the geometrical circle of least confusion, we find the results in Table III. The central intensity is only ·275 of that in the ideal image, but it drops to about one-third of the central value (instead of one-half in the ideal disc) for the phase-angle of 90°. Direct observation with microscopic objectives, which lend themselves to the introduction of definite amounts of spherical aberration by a change of tube length, demonstrates that far greater reductions in size of the central condensation are obtainable, apparently with considerably *larger* amounts of spherical aberration. It is hoped that the proper proportions will be determined in the near future.

It is highly interesting to ascertain what amount of longitudinal spherical aberration in the geometrical sense corresponds to the marginal differences of phase to which the above integrations refer. We found that the equivalent separation between the marginal and paraxial foci is that corresponding to an r^2 difference of phase equal to twice the spherical aberration (r^4 term) present. In the footnote on p. 582 it was shown that for a ratio of aperture to focal length of 1 : 15 one wave-length of difference of optical paths due to change of focus corresponds to ″·038 movement of the eyepiece. Therefore the $\frac{1}{4}$ wave-length of marginal spherical aberration which was shown to be harmless and almost indistinguishable corresponds for telescopes of the stated aperture-ratio to a longitudinal spherical aberration of ″·019, whilst the whole wave-length of spherical aberration, which corresponds to the Rayleigh limit at the best focus, and only leads to a loss of 20 per cent. of light from the central star-disc, is equivalent to a geometrical aberration of ″·075, a surprisingly large amount, which, however, is fully borne out by extensive practical experience in the testing of imperfect optical systems. In systems of a lower ratio of aperture to focus the figures change, of course, inversely as the square of the ratio, in accordance with the formula given in the footnote. In reflectors of a 1 : 5 ratio the permissible longitudinal aberration thus goes down to $\frac{1}{9}$ of the above figures.

Higher Spherical Aberration.

As a rule, the ordinary spherical aberration is *corrected* (at any rate on paper) by the optical designer. Instruments in which all curvatures of surfaces and convergence angles of the marginal ray are small will then be sensibly free from aberration and should give the Airy image. But when deep curvatures and large angles occur in a system, as is the case in photo-visual objectives, in ordinary photographic objectives, and especially in

microscope lenses, then the terms in r^6 and still higher ones become important, and we get secondary wave-distortion. At the focus of the paraxial rays the difference of optical paths will then, in second approximation, be expressed by the law

$$\phi = c_2 r^4 + c_3 r^6.$$

When such a system is calculated by the first approximation analytical formulæ the r^4 term will vanish and the r^6 aberration will be left in, and we shall have the wave-distortion expressed by

$$\phi = c_3 r^6.$$

In focussing the instrument we shall again try to balance this by r^2 distortion, obtained by a suitable change of focus. It is easily shown that the residuals will be a minimum at the focus corresponding to

$$\phi \text{ (minimum)} = c_3 r^6 - c_3 r^2.$$

This attains its highest value at $\sqrt[4]{\frac{1}{3}}$ of the full aperture, for which $\phi = -\frac{2}{3}\sqrt{\frac{1}{3}}c_3$ or $-\cdot385c_3$. The Rayleigh limit will therefore be represented by an amount of r^6 aberration of $\cdot25/\cdot385 = \cdot65$ wave-lengths.

In the *trigonometrical* calculation of the same system the marginal ray will be brought to the focus of the paraxial ray. As was shown above, this corresponds to making $d\phi/dr = 0$ for the marginal ray, for which we again take $r = 1$. Differentiating $\phi = c_2 r^4 + c_3 r^6$ and determining c_2 in terms of c_3, we thus find that trigonometrical calculation leads to the formula of the wave-distortion at the paraxial focus

$$\phi = c_3 r^6 - \tfrac{3}{2} c_3 r^4.$$

For the marginal zone this gives a difference of optical paths of $\frac{1}{2}c_3$. This can be directly computed by the formula given in my paper (*Monthly Notices*, 1905 April, 65, 603) on spherical aberration, and the actual amount of the higher aberration present can thus be estimated.

In focussing such an instrument, we shall again depart from the paraxial focus in order to find the brightest star-disc: an r^2 term thus comes in. A somewhat more complicated solution of this minimum problem leads to the equation

$$\phi \text{ (minimum)} = c_3 r^6 - \tfrac{3}{2} c_3 r^4 + \tfrac{9}{16} c_3 r^2,$$

which reaches the same highest value $= \frac{1}{16}c_3$ as a maximum for $r = \frac{1}{2}$, and as the limiting value for $r = 1$, and is zero for $r = 0$ and for $r = \sqrt{\frac{3}{4}}$. It will be seen that the trigonometrical correction is enormously superior to the analytical one, for the residual distortions are in the proportion of $\cdot385$ to $\cdot0625$ or as $6\cdot16$ to 1!

It has been assumed that the coefficient of the r^6 term remains the same when the slight change of curvatures is made which

distinguishes the trigonometrical result from the analytical one. This is sensibly correct, but any change in c_3 will almost invariably be again in favour of the trigonometrical process, because it usually leads to a slight flattening of the deepest curves at which the bulk of the higher aberration arises.

This means of course that a given system, the aperture of which is limited by higher spherical aberration, when computed trigonometrically can be given a clear aperture $\sqrt[6]{6{\cdot}16}$ or 1·35 times as large as would be admissible for a similar system computed by analytical first approximation formulæ.

As only $\frac{1}{16}$ of the r^6 aberration appears as wave-distortion at the best focus of a trigonometrically computed system, the r^6 aberration may reach four whole wave-lengths before the Rayleigh limit is attained.

I accordingly have carried out this integration for the formula of the phase-angle in degrees,

$$\phi = 1440^\circ\, r^6 - 2160^\circ\, r^4 + 810^\circ\, r^2,$$

with the results tabulated in the last section of Table III. It will be seen that the central intensity is lower than in the other cases of the Rayleigh limit: it is only ·73 of the ideal maximum, but the curious proportionality of the fall of intensity to that in the Airy disc is once more well maintained, and consequently also the resolving power. A very remarkable feature of this diffraction pattern corresponding to $\frac{1}{4}$ wave-lengths of "zonal" aberration is, that the first ring is actually *less* intense than that of the ideal Airy image; the light lost in the central disc must accordingly be looked for in the outer rings; and it will in fact be seen from the figures that there is considerable luminosity where the Airy image has its second dark space, and that the *second* ring of the "zonal" image is considerably brighter than the *first* ring of the Airy image.

When the higher aberrations become still heavier, good results can only be hoped for when the sixth order term is also brought under control; we can then establish a favourable balance for the eighth order term. We shall have ϕ expressed by an equation of the form

$$\phi = c_2 r^4 + c_3 r^6 + c_4 r^8.$$

In my paper on spherical aberration, already quoted, it was stated that the higher spherical correction should take the form of bringing the marginal ray to the focus of the paraxial rays, and of causing it to arrive there in the same phase of vibration as the axial ray. This gives the two conditions

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dr} = 0 \text{ for } r = 1.$$

From the two resulting equations c_2 and c_3 can be determined in terms of c_4, and we obtain the equation for

$$\phi \text{ (minimum)} = c_4(r^4 - 2r^6 + r^8).$$

This is zero for $r=0$ and for $r=1$, and has a maximum value $=\frac{1}{16}c_4$ for $r=\sqrt{\frac{1}{2}}$. If we carry out the correction for the real marginal ray of the system we shall therefore have a zonal wave-distortion in this region of $\frac{1}{16}$ of the r^8 aberration present. But we can do better than that. The zero value of the expression for ϕ at $r=1$ is a true minimum; therefore the wave, beyond $r=1$, bends away again to the same side of the ideal spherical wave as that of the maximum at $r=\sqrt{\frac{1}{2}}$. It is easily found that it reaches the value of that maximum ($\frac{1}{16}$) at $r=1{\cdot}10$. Consequently we can open out the aperture to $1{\cdot}1$ times that of the calculated marginal ray without increase of the residual zonal distortion; we merely obtain it in two different regions. Now at $r=1{\cdot}1$ the r^8 terms will be $1{\cdot}1^8$ or $2{\cdot}144$ times as large as at $r=1$, and we therefore find that the zonal distortion will be only $\frac{1}{3}\frac{}{4}$ part of the r^8 aberration if we calculate the "marginal" ray through $\frac{10}{11}$ of the full aperture. I have not yet carried out the numerical integration for this case, but the best wave-form which we have found for the case of eighth order aberration resembles the corresponding one for sixth order aberration so closely that we may safely anticipate a very close agreement in the forms and intensities of the spurious discs.

I have gone yet a step higher in the correction of microscope systems of high numerical aperture without having to compute additional rays. In the paper on spherical aberration the formula for the difference of optical paths was also reduced to its first order (r^4) term. Adding the condition that this shall also be zero, we remove the r^4 term from the equation of the wave-form and can gain control over the r^{10} term by establishing a balance of the terms in r^6, r^8, and r^{10}, with a still higher freedom from residual aberrations. But I do not propose to elaborate this case in the present paper.

The striking and highly important result of this discussion of aberrations of successively higher orders is that the permissible amounts which may be left uncorrected rise steadily with the order. $\frac{1}{4}\lambda$ of r^2 distortion (departure from true focus) brings us to the Rayleigh limit. Of ordinary spherical aberration (r^4) we can admit a whole wave-length. When we come to the sixth order 4 wave-lengths are permissible. And with eighth order aberration the Rayleigh limit is compatible with $8\frac{1}{2}$ wave lengths of this term, always provided that we really have control over the successive terms in accordance with the conditions laid down in this paper.

This steadily growing tolerance in the higher aberration is the true and only explanation of the possibility of lens-systems in which the angles of incidence approach uncomfortably closely to their natural limit of 90°.

Some caution must, however, be used in taking advantage of the results indicated; they would be strictly true if the series of aberrations really broke off with the last term included in each of the successive equations for ϕ. In reality, higher terms will be

always present, and will become amalgamated with the highest included in the equation, with the easily estimated result of a less favourable magnitude of the residual distortion. I had in fact adopted limiting values for these residuals corresponding to half the Rayleigh limit long before the detailed theory had been worked out, chiefly on the basis of trials of a few experimental microscope objectives with definitely known amounts of the residual aberration.

But it will be clear from the results of the integrations given in this paper that whilst the classical Rayleigh limit is abundantly justified as far as resolving power is concerned, there is in every case a decided loss of contrast in the image of extended objects, and I therefore venture to suggest that for instruments of the highest quality the limit should be reduced to $\frac{1}{6}$ or even $\frac{1}{8}$ of a wave-length whenever this degree of perfection can be reached.

Coma.

The method of integration adopted in this work is applicable to the elucidation of another important and much discussed problem, viz. the form of the image of a star moderately removed from the optical axis when coma is present, and its location with reference to the geometrical coma-figure. It is a very important question, because it is often necessary or convenient not to fulfil the sine-condition exactly; in fact the vast majority of astronomical object-glasses do not fulfil it. How far may we go in this offence against the sine-condition without serious deterioration of the extra-axial image? On the present occasion I propose to give only a brief sketch of my work on this subject, as the calculations, which are decidedly laborious, have not yet been completed.

In my paper on the Sine-Condition (*Monthly Notices*, 1905 March, **65**, 506) it was proved that in any centred optical system a point can always be found in the image-space in which all the light passing from an extra-axial object point through a narrow zone of the system, concentric with the optical axis, arrives in the same phase of vibration and thus produces a maximum of intensity. Fulfilment of the sine-condition in the absence of spherical aberration causes identity of this image-point for all zones, and therefore the formation of a perfect Airy disc for extra-axial image-points. If the sine-condition is not fulfilled, then each zone has a different image-point, and the totality of these points covers a radial line in the field of view, limited at one end by the point of best concentration of the light from the centre of the optical systems, at the other by that from its extreme marginal zone. Geometrical tracing of the *rays* passing through the same zones results in the well-known Coma-figure (see my paper on "The Five Aberrations of Lens-Systems," *Monthly Notices*, 1918 November, p. 65) built up of a succession of circles of growing diameter, over the circumference of which the rays from successive zones are uniformly distributed in pairs. It is easily shown that the image-

points arrived at by the sine-condition are identical with the points of the corresponding coma-circles nearest to the head of the coma-figure. We thus arrive by the sine-condition at the straight line HT_s, fig. 3 of the present paper, as containing all the points in which the light from successive zones arrives without difference

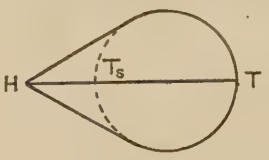


FIG. 3.

of phase, whilst according to geometrical ray-tracing the light is distributed over the whole area of the complete coma-figure HT, but with a strong concentration towards its head H. There is thus a disagreement to the extent of a 1 to 3 ratio as to the length of the coma-figure by the two methods.

In both cases there is agreement as to the law according to which the successive image-points or circles move away from the point H with increasing diameters of the zones. The distribution is according to the square of the diameter of the zone, or proportional to our integration-variable r^2. Consequently the maximum concentrations of the light of our successive zones are equally distributed over the length HT_s, and as there are no differences of phase to be considered (such as we met with in dealing with spherical aberration), the values of successive zones of a table like Table II., but with suitably varied intervals of the W argument, have only to be arranged with a horizontal shift of successive lines by one column always in the same direction, thus producing a diagonal arrangement, in order to effect the integration by summation of the vertical columns which thus result. Several such integrations have been carried out, with the result that when the length of the line HT_s corresponds to 160° of our W scale, or to $\frac{8}{9}$ of the classical Dawes limit of resolution (*i.e.* one second of arc for $4\frac{1}{2}$ inches aperture), there results a spurious disc with a central intensity equal to ·89 of the ideal maximum, and with very nearly uniform falling off of this intensity in all azimuths and in proportion to that of the Airy disc. There is more want of symmetry in the rings; these are replaced by a gradual falling off in intensity in the direction towards the head H of the coma-figure, and are more pronounced than the Airy rings in the opposite direction. But with so strong a concentration of light in the central disc this slight anomaly is quite unimportant, and we may say that such an amount of coma is practically harmless in every respect. The point of greatest concentration of the light is found near $\frac{2}{3}$ of the distance from H to T_s or at about $\frac{2}{9}$ of the distance from H to T. We may therefore safely say that there will be no similarity at all between the actual light distribution and the geometrical coma-

figure. With a somewhat higher amount of coma (208° of the W scale) the central image, and especially the rings, become a little more unsymmetrical, again with the more gradual falling off towards the head of the coma-figure—in exact contradiction of the geometrical result—and the centre of gravity of the central disc shifts towards the head of the figure. The maximum intensity is ·83 of the Airy maximum, and the diameter of the central condensation differs very little from that in the Airy image.

With heavy coma (640°) the image shifts to a position at less than half the distance HT_s from H and becomes very decidedly spread out, but still retains a considerable approach to symmetry in the central condensation.

These results confirm an old experimental conclusion, arrived at by tests of microscope objectives deliberately computed so as to offend against the sine-condition to a known extent, to the effect that the magnification as determined by the sine-condition which is produced by the marginal zone of an optical system may differ from that of the paraxial zone to the extent of 1 part in 400 without notable effects on the perfection of the image within the field of any eyepiece *mounted and kept in the optical axis.* Instruments in which the eyepiece can be moved in a plane at right angles to the optical axis would of course call for a closer fulfilment of the sine-condition.

Imperial College of Science and Technology:
1919 *June.*

On a new Artificial Horizon for Sextants. By Dr. L. Becker, Regius Professor of Astronomy in the University of Glasgow. (Plate 17.)

1. The old artificial horizon is, in point of accuracy, superior to any of the modern devices by which the altitude may be measured with the sextant. Essentially it consists of a mirror controlled by gravity which might either be horizontal or inclined to the horizon and in that case attached to the sextant. This artificial horizon furnishes an auxiliary image, A.I., which is brought into coincidence with the sextant image, S.I., of the same object, and it is of no consequence at what part of the field the coincidence is observed. At coincidence the sextant's plane is bound to be vertical, and there is no risk that an angle in a wrong plane be measured. It is, however, a serious disadvantage that the instrument cannot be employed when the altitude of the object is small or exceeds 60°, and it is an inconvenience that the inclination of the line of sight changes with the altitude of the object.

Nothing is altered in the general properties of the images by the addition to the instrument of an even number of mirrors relatively

fixed, and while the advantages of the old artificial horizon will be maintained, it may be feasible to extend its applicability.

In the instrument discussed in this paper the rays from the object which give the A.I. are twice reflected on mirrors relatively fixed before they fall on the mirror controlled by gravity. I choose as one of these mirrors the index-glass of the sextant, with

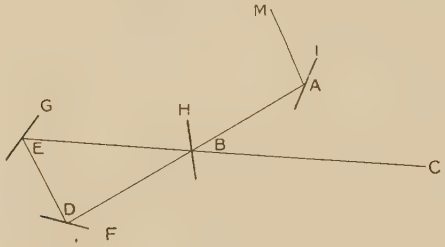


FIG. 1.

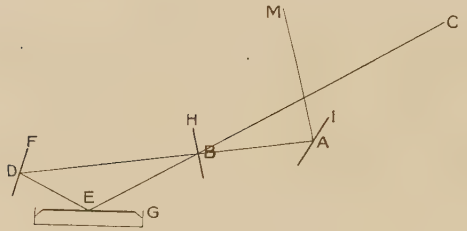


FIG. 2.

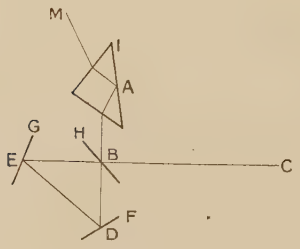


FIG. 3.

the effect that for all altitudes of the object the line of sight obtains a constant inclination to the horizon.

2. *The new Artificial Horizon.*—The arrangement of the mirrors is illustrated in figs. 1, 2, and 3. The second diagram belongs to a horizontal mirror, and the third to a prism in place of the index-glass. In the diagrams I stands for the index-glass, H for the horizon-glass, F for the additional fixed mirror, and G for the mirror controlled by gravity. The course of a ray MA coming from the object, M, is along MABC for the S.I. and along

MABDEBC for the A.I. The index-glass of the sextant is large enough to send light through the transparent portion of the horizon-glass, which light is unused in the ordinary sextant.

Let, in fig. 4, $\frac{1}{2}$R denote the angle of inclination of the two sextant mirrors, δ the constant angle of inclination of the H- and F-mirrors, g the constant zenith-distance of the normal to the G-mirror, i the inclination of the line of sight, OC, with reference

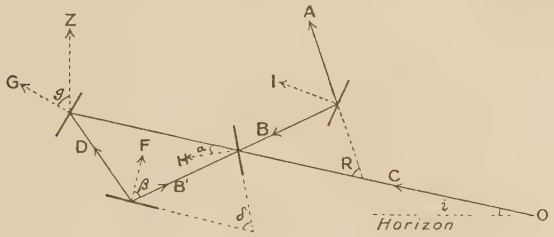


Fig. 4.

to the horizon, α and β respectively the incident angles at the H- and F-mirrors, when the two images coincide; Z the zenith; and h the altitude of an object. The plane of drawing represents a vertical plane. The angle enclosed by the rays at the G-mirror being $2(90^\circ - g - i)$, the diagram gives

(1) $$i = \delta - g,$$

and

(2) $$h = \mathrm{R} + i.$$

Therefore, whenever the A.I. coincides with the S.I., at a vertical position of the sextant's plane, the line of sight has a certain constant inclination, i, to the horizon, and the altitude of the object equals the sum of the reading of the circle and a constant index-error. This index-error differs from that of the ordinary sextant by the constant inclination, i.

On the assumption that the mirrors have been so adjusted that their normals lie in the same plane, and that this plane when vertical also contains the normal to the G-mirror, coincidence of images can only occur in the vertical plane passing through the object if the G-mirror is properly fixed to the sextant's frame. It is therefore necessary that not only the inclination of that mirror must change with reference to the sextant's plane when the latter is turned round a horizontal axis, but the mirror must also participate in the turning of the sextant round a vertical line. The mirror might, for instance, be attached to a pendulum hung in gimbals which are supported by the frame of the sextant, provided that the plane of their axes does not contain the normal to the mirror. Further, a horizontal mirror need not be attached to the frame, and a vertical mirror will not give the desired result. These deductions also apply to the old artificial horizon.

3. *Errors of Instrument.*—I assume that the normals to the two mirrors of the sextant lie in the sextant's plane, which is the plane at right angles to the axis of rotation of the index-glass. The new fixed mirror, F, and the gravity mirror, G, may be out of adjustment. According to No. 2, the mirror, G, controlled by gravity cannot turn round a vertical axis relatively to the sextant when the sextant is held in the position at which the images fall together. Hence in this position, and for all azimuths of the pole, S, of the sextant, the azimuth of the normal, G, to the gravity mirror exceeds the azimuth of S by a constant quantity which I designate by $90^\circ - \gamma$. The inclination, f, of the plane of reflection at the fixed F-mirror with reference to the sextant's plane and γ are the errors of the instrument. The altitude of an object must be a function of the errors f and γ.

Notation employed in Figs. 4 and 5.—Z is the zenith, I, H, F, G

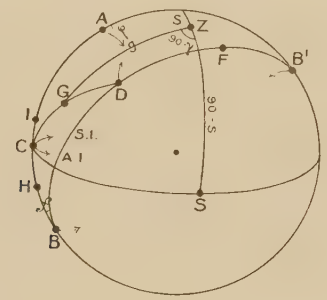


FIG. 5.

are respectively the normals to the four mirrors counted in the directions indicated in fig. 4.

$h = 90^\circ - AZ$ is the altitude of the object, A, in fig. 5.

$\frac{1}{2}R$ is the angle of inclination between the mirrors I and H. It alters with the turning of the index. R also equals AC, and it is the reading of the circle, corrected for index-error of the ordinary sextant.

$\delta = 180^\circ - HF$ is the constant angle of inclination between the H- and F-mirrors.

$g = GZ$ is the constant zenith-distance of the normal to the G-mirror.

$\alpha = CH = HB$ is the incident angle at the H-mirror.

$\beta = B'F = FD$ is the incident angle at the F-mirror. Both angles α and β vary, to the extent of a degree, with the position of the images in the field of view of the telescope.

$i = 90^\circ - ZC$ is the inclination of the line of sight with reference to the horizon.

$s = 90^\circ - ZS$ is the altitude of the pole, S, of the sextant.

For an approximately adjusted instrument f and γ are small quantities of which the cubes are less than a second and always negligible. Fig. 4 shows the path of the rays for zero errors, and

fig. 5 shows on a sphere the directions of the rays and normals to mirrors, etc., when the errors are not zero. The points entered on the sphere correspond to the directions indicated by arrows in fig. 4. The two images, the sextant-image S.I. and the auxiliary image A.I., coincide at C.

By triangles ZAS and ZAC

$$(3)\quad \left.\begin{matrix} h \\ 180^\circ - h \end{matrix}\right\} = \mathrm{R} + i + \tfrac{1}{2}s^2 \tan i \mp \tfrac{1}{2}s^2 \tan h + \ldots, \quad \mathrm{R}\left\{\begin{matrix} \angle \\ Z \end{matrix}\right. 90^\circ - i.$$

By triangle HFB

$$(4)\quad \delta = \alpha + \beta - \frac{\sin\alpha \sin\beta}{2\sin\delta} f^2 + \ldots$$

By triangles ZCS, ZCG, and BCD

$$(5)\quad i = \alpha + \beta - g + \frac{\cos i \sin g}{2\cos(g+i)}(\gamma + s\tan i)^2 - \frac{\sin 2\alpha \sin 2\beta}{4\sin 2(g+i)} f^2 + \ldots$$

By triangles GSC, GSZ, and BCD

$$(6)\quad s = \frac{\sin 2\beta}{2\cos g \sin(g+i)} f - \gamma \tan g + \ldots,$$

and hence

$$(7)\quad i = \delta - g + \left(\frac{\sin^2 2\beta \cos i}{4\sin\delta \sin 2\delta \sin g} - \frac{\sin^2\alpha \sin^2\beta}{\sin 2\delta}\right) f^2 + \frac{\cos\delta}{2\sin g \cos i} s^2 - \frac{\sin 2\beta}{2\sin g \sin\delta} fs.$$

$$(8)\quad \left.\begin{matrix} h \\ 180^\circ - h \end{matrix}\right\} = \mathrm{R} + \Delta\mathrm{R}, \qquad \mathrm{R}\left\{\begin{matrix} \angle \\ Z \end{matrix}\right. 90^\circ - i.$$

The altitude, s, of the pole of the sextant at which the two images are in coincidence is a function of the constant errors, f and γ, according to (6), and it has a constant value of the order of the errors when the coincidence occurs near the centre of the field. For coincidences near the edge of the field the value of s differs from this constant by quantities of the second order on account of the slight variability of β.

The inclination, i, of the telescope with reference to the horizon at which the A.I. coincides with the S.I. differs from the constant value of $\delta - g$ by constant quantities of the order of the squares of the errors and variable quantities of the third order.

Thus coincidence of the images is an indication that the sextant's plane is held at a small constant inclination to the vertical, this inclination being of the order of the errors in adjustment, and that the line of sight is placed at a constant inclination to the horizon. The altitude of the object above the horizon differs from the reading of the circle by a constant index-error, $\Delta\mathrm{R}$, if squares of errors be neglected.

4. *Effect of turning of Sextant's Plane.*—The result obtained in the preceding paragraph for coinciding images may be extended to images standing on a horizontal line. Let the plane of the sextant be turned round the direction towards the object, A, through an angle θ. This rotation does not change the relative position of the points A, I, H, B, B′, D, C, S, which move into A, I_1, etc., while it alters the position of this configuration with reference to the zenith, Z. The S.I. will stand at C_1. The rotation round A may be replaced by its two components, $\theta \cos z$ round Z and $\theta \sin z$ round a horizontal axis. The gravity mirror which is free to turn round any horizontal axis only turns round Z, from G to G_1, in fig. 6. The difference in azimuth of G_1 and S_1 is again $90° - \gamma$, and the zenith-distance of G_1 is g. The A.I. stands at C_1' where $C_1'G_1 = G_1D_1$.

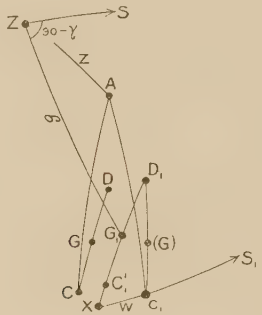


FIG. 6.

The great circle S_1C_1, which is at right angles to C_1A, cuts D_1C_1' at X. Designate C_1X by w. With regard to only first-order terms in f and s_1, the A.I. stands at distance w from the S.I. and on a horizontal line. Let (G) lie half-way between C_1 and D_1, then by diagram, fig. 6, $G(G) = GG_1 + G_1(G) + \ldots$ or

$$\theta \sin(g - z) = \theta \cos z \sin g + w \cos(\beta + \alpha)/\sin(2\beta + 2\alpha) + \ldots,$$

hence

$$\theta \sin z = -w/(2 \cos g \sin \delta).$$

Let s_1 designate the altitude of the sextant's pole which is not affected by the rotation round the zenith, then

$$(9) \qquad s_1 - s = \theta \sin z = -w/(2 \cos g \sin \delta).$$

$G_1(G)$ subtends at Z an angle which equals $w/(2 \sin g \sin \delta)$, so that the difference in azimuth of (G) and S_1 is $90 - \gamma - w/(2 \sin g \sin \delta)$. The zenith-distance, g_1, of (G) differs from g only by second-order quantities, viz. the product of G(G) into the angle subtended at G by AZ. All the formulæ, therefore, given in the last paragraph also belong to this case if γ be changed into $\gamma + w/(2 \sin g \sin \delta)$, i into i_1, s into s_1. It is thus proved that, apart from quantities of the order of squares of w and of the errors, the inclination of

the line of sight has the same value as for coincidences, and the altitude is given by formula (8). Accordingly, when taking observations it is quite sufficient to place the two images on a horizontal line instead of making them coincide, provided that the images are not more than $\frac{1}{2}^\circ$ apart for moderate altitudes.

5. *Curve described by the A.I. round the S.I. when the Observer turns in Azimuth.*—Starting from the coincidence of the two images which occurs when the altitude of the sextant's pole is s, turn the sextant's plane round the direction towards the object, and place the S.I. and A.I. on a horizontal line by turning the index. The reading is R, the distance of the images is w, and the azimuth and altitude of the pole are $A_1 - 90^\circ$ and s_1. When the index is turned to R_0 the two images separate by $2(R - R_0)$. The co-ordinates of A.I. relatively to S.I. are $x = 2(R - R_0)$ and $y = w$, the X-axis being directed vertically upwards in the field of view and the Y-axis towards the left. But $w = c(s - s_1)$ by (9), and s_1 and R can be eliminated by the azimuth, A, and the zenith-distance, z, of the object. Triangle AZS_1 gives

$$s_1 + \ldots = \tan z \sin (A - A_1),$$

and

$$\sin (90^\circ - R - i_1 + \ldots) = \sin z \cos (A - A_1).$$

According to the first formula, either z or $A - A_1$ are small for the points of the curve lying in the field of view. Let the arbitrary R_0 be the value of R which belongs to $A_1 = A$, *i.e.* to a vertical sextant plane, then for small z

$$x - 2z = -2z \cos (A - A_1), \qquad y - cs = -cz \sin (A - A_1),$$

and for small $A - A_1$

$$x = \tan z\,(A - A_1)^2 \qquad y - cs = -c \tan z (A - A_1),$$

where $c = 2 \cos \varphi \sin \delta$.

Fig. 7 shows, for $c = 0{\cdot}7$ and two values of the zenith-distance, the conic sections described by the A.I. relatively to the S.I. which stands at C if the index be clamped at R_0. Coincidence of the two images belongs to the intersection of the conic section with the X-axis, because the x-co-ordinate can be made to vanish by the turning of the index.

The ellipse does not intersect the X-axis for small zenith-distances, $z < s$, where s is a constant of the order of the errors i and γ, and therefore, in that case coincidences are impossible, and the images must be observed when they stand on a horizontal line. On account of the term $\tan h s_1^2$ in (3) which now is of the order of the errors and cannot be neglected, s_1 must be made zero, *i.e.* the sextant's plane must be vertical. According to the diagram the A.I. stands then at the lowest or highest point of the orbit.

In this exceptional case of a very small zenith-distance the observer would clamp the index-glass when both images appear and turn with the instrument in azimuth until the A.I. is seen near the lowest or highest point of the orbit, and then turn the

index, thus placing the two images on a horizontal line. The altitude is given by (8).

But generally the zenith-distance will be much greater than s, and the curve will intersect the X-axis at M. The A.I. can be brought to M by turning in azimuth, and the images can be made to coincide at M by turning the index. The altitude is again given by formula (8), because in (3) the term $\frac{1}{2}\tan hs^2$ is now of the second order and negligible. Instead of the coincidence the lowest point of the orbit might have been chosen as before, and

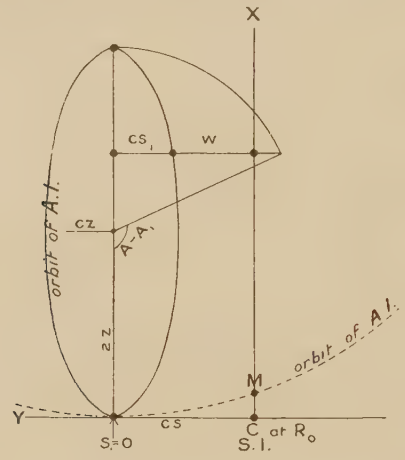


FIG. 7.

the two images be placed on a horizontal line. Formula (8) would then strictly be correct. The readings in the two cases differ by $\frac{1}{2}MC = \frac{1}{2}\tan hs^2$. MC thus represents the greater portion of the quantity neglected when formula (8) is applied to coincidences.

6. *The Relative Positions of the S.I. and A.I. of a Straight Line.*—Let the arrow shown at A in fig. 5 as a short arc of a great circle represent a straight line. Its inclination to the plane of the sextant is designated by ϕ. The position of the arrow after each reflection is drawn at the points B, C, B′, D, C. At C both images are again great circles, the S.I. standing at position-angle ϕ and the A.I. at angle $180° - \phi + f - D + 180° - C$. C and D are angles in the triangle CDB and can be expressed by the error f of the instrument. To obtain the position of the images in the field of view, draw lines diametrically opposite to the images shown at C and view the sphere from outside. In the field of view the images appear as straight lines, the S.I. at position-angle ϕ, and the A.I. at position-angle $\phi + \psi$, where

$$\psi = 180° - 2\phi + 2f \cos\alpha \sin\beta / \sin\delta.$$

The position-angles are counted anti-clockwise, *i.e.* in the same direction as for the object, from a point at the bottom of the field.

Compared with the object, the S.I. is turned 180° and the A.I. is reversed right and left, but not up and down, and turned through an angle $2f \cos\alpha \sin\beta / \sin\delta$. Fig. 8 gives the naked-eye view of an object and fig. 9 the appearance of the images in the field of view.

Fig. 8.

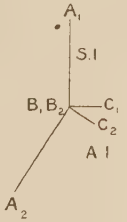


Fig. 9.

It was assumed that the gravity mirror is so adjusted that there are two coinciding images. Its position relatively to the zenith does not enter the discussion, and therefore the result holds good when the gravity mirror is clamped relatively to the sextant's plane and the plane turned into any position.

7. *Adjustment of Ordinary Sextant.* — The index-glass and horizon-glass are adjusted by the well-known methods.

As to the adjustment of the telescope, I define the sextant's equator as the line in the field of view in which a plane drawn through the centre of the object-glass at right angles to the mirrors intersects the field of view. All observations made with or without the artificial horizon ought to be taken on or near to this equator.

The position of the sextant's equator can be found with the aid of the direct image, D.I., and the S.I. of a straight line, and it is advantageous to fix the sextant on a stand. When the line is at a short distance from the sextant there is perceptible parallactic displacement of the S.I., and for the parallel position of the two mirrors the D.I. and the S.I. will not coincide unless the image lies in the equator, or, in practice, intersects the equator at a point in the field of view and at a small inclination. The images are made to fall together by turning the sextant round an axis which lies in the sextant's plane at right angles to the telescope.

The second component of the rotation of the sextant's plane is obtained by comparing the S.I. of another part of the line at reading R with the former D.I., and the images are brought together by turning the sextant round the telescope. The images now indicate the position of the equator. The line may be a plumb-line, and in that case the sextant's plane will be vertical.

8. *Adjustment of the Mirrors of the Artificial Horizon.*—Without disturbing the position of the sextant found by the method described in last paragraph, attach the G-mirror to the sextant, and inspect the S.I. and A.I. of the line. If they do not lie parallel to each other, the normal to the F-mirror does not lie in

the sextant's plane but is inclined by f, and its inclination, f, must be altered in accordance with the results derived in section 6.

The sextant's plane is made vertical either by mechanical means or by using a plumb-line in the method of last paragraph. The S.I. and A.I. of a distant object are made to coincide by turning the G-mirror in azimuth. This adjustment makes γ vanish.

9. *Determination of Index-Error.*—The index-error ΔR must be determined by means of the equations No. 8 in section 3. One measurement suffices for its determination if the altitude of the object be known.

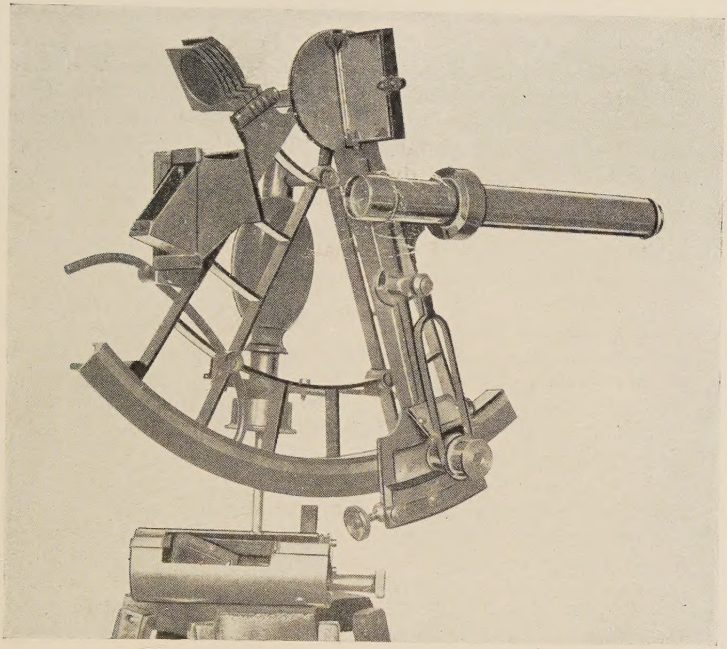
The ordinary sextant permits of the measurement of an elevation of about 120° above the line of sight, and therefore an object within 30° from the zenith may be observed in two ways with the new instrument at a horizontal direction of the telescope, either with the line of sight in the vertical of the object or 180° in azimuth from it. In the first observation the observer faces the object and in the second his back is turned towards it. h and ΔR can then be calculated from the two equations, No. 8.

Another inclination might be chosen for the line of sight, and thus the range of zenith-distance increased, for which two observations are possible.

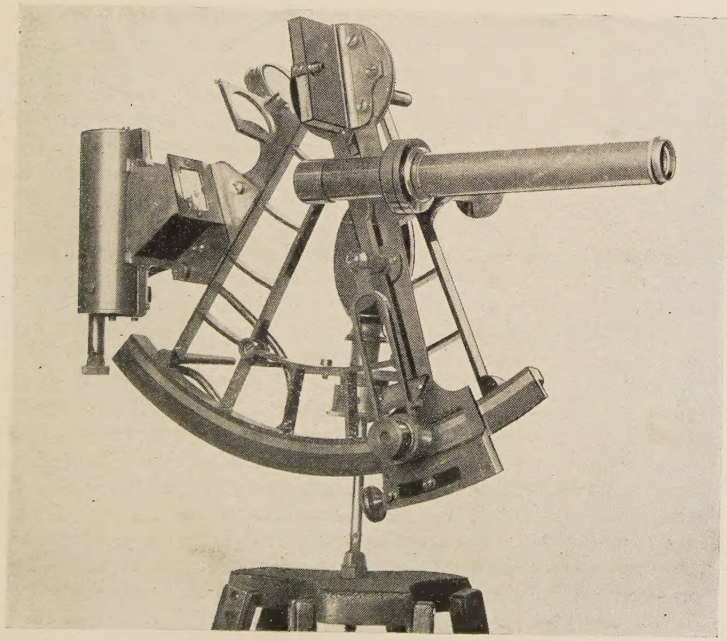
Or the index-glass of the sextant may be replaced by a right-angled prism and the course of the rays may be arranged as drawn in fig. 3. With such a "prism circle" all angles from 0° to 180° can be measured, and two observations can be taken of any object in the opposite verticals.

Though one may, as a rule, depend on the constancy of the index-error, it is nevertheless desirable that the effect of an erroneous value be eliminated in the event of the above methods not being applicable. At night two stars ought to be observed instead of every star otherwise required, the two stars lying at opposite sides of the zenith, either near the meridian or prime vertical. A curve drawn midway between the Sumner's lines belonging to two such observations reduced with a wrong value of the index-error can be proved to pass very nearly through the position of a moving ship at the mean of the times of observation. The error is not 0′·5 for an error of 20′ in the index-error, on the assumption that the stars lie within 20° from either the meridian or prime vertical, and more than 35° from the zenith, with an interval of not more than five minutes between the observations.

10. *Description of Instrument.*—The photographs (Plate 17) show an ordinary sextant in which the horizon-glass has been replaced by a small box which contains the two fixed mirrors, viz. the horizon-glass and the new F-mirror, the former being visible in the second photograph and the latter in the first. The mirror controlled by gravity is attached to a pendulum hung in gimbals and contained in a tube so as to be protected against wind pressure. In the first photograph the tube lies below the sextant, which may thus be used in the ordinary way, and it is attached to the mirror-case in the second photograph.



1. Sextant with Mirror-Case, for Ordinary Sextant Observations.



2. Sextant with Pendulum attached to Mirror-Case, for Observations with Artificial Horizon.

A special feature of the instrument is the rotating stop, which when brought into action prevents the pendulum from swinging in the direction towards the observer. It is acted upon by the lever visible in both photographs.

11. *Method of Observing.*—The S.I. is found in the usual way. The A.I. does not require to be specially searched for. It will be seen near the S.I. when the pendulum is arrested by the stop. While the observation is being carried out the two images are kept approximately on a vertical line by turning the sextant's plane, with the effect that that plane is always sufficiently near the vertical. On releasing the pendulum the A.I. will move out of the field. The observer merely notes the direction in which the A.I. moves and correspondingly turns the index. This procedure, viz. arresting and releasing of pendulum and turning of index, is repeated until the A.I. remains in the field and begins to oscillate with a free pendulum. The S.I. is then brought approximately to the centre of the orbit of the A.I. by turning the index.

By these preliminary steps the index is nearly turned into the correct position, and if the oscillations were damped the two

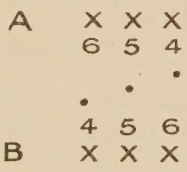


Fig. 10.

images would nearly coincide. When the instrument is held by hand the oscillations may become very large after some seconds and require to be stopped for an instant by bringing the stop into action.

Fig. 10 explains how the observations are carried out when the instrument is held by hand, the dot representing the S.I. and the crosses the highest and lowest points, A and B, of the orbit of the A.I. The index is set so that the S.I. stands below the centre of the orbit when the object stands west of the meridian. The S.I. alters its position relatively to the orbit on account of the rotation of the celestial sphere, and the time is noted when the S.I. divides the amplitude AB in the proportion 4 to 6, and again when the proportion is 6 to 4. The mean of the two figures gives the time belonging to the reading of the circle.

12. *Error of Observation.*—Fig. 10 gives an idea of the order of the error that might be expected in such observations. The observer will easily appreciate the difference between the proportions 3 to 7, 4 to 6, 5 to 5, etc. Indeed, in transit observations by the eye-and-ear method, which are similar to the above estimates, the error is not a tenth of the range AB but only a fourteenth (0·07 second). Since a change in the circle reading alters the relative distance of the S.I. and the A.I. by twice that

amount, the above differences are equivalent to a change in the reading amounting to AB/20. An observation contains two estimates of the interval, and therefore the error of an observation may be of the order of AB/28, *i.e.* 2′ for an amplitude of a degree. In the actual observations the pendulum is released for only about two seconds at a time, during which there are eight oscillations of the A.I. and the amplitude is generally less than a degree. Though the motion is erratic the centre of the space over which the A.I. moves is nearly the position of the A.I. when the pendulum is in equilibrium, and therefore the points A and B are the extreme positions during the one or two seconds. The error actually found is less than the above figure, viz. 1′. It is the average error of numerous observations I have taken on land, holding the instrument by hand. I have used the instrument at sea on a day when the steamer was moderately rolling, and though I had no means of checking the result, the observations did not appear to me to be more difficult to execute than on land.

At sea there is another source of error arising from the horizontal acceleration of the instrument. This error can, however, be avoided to a great extent by properly timing the observations, for its effect will disappear in the mean of observations taken either at the position of equilibrium of the ship or equally far from it on both sides.

On land a light camera-stand should always be used with the instrument, and the instrument might advantageously be the prism-circle described in paragraph No. 9. Numerous observations made with the instrument on such a stand give an average error of 12′ for one observation.

Aviemore: 1916 *June.*

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